Intercepts, Asymptotes, and Discontinuity of Functions

Strand: Functions

Topic: Exploring asymptotes and discontinuity

Primary SOL: AII.7 The student will investigate and analyze linear, quadratic, absolute value, square root, cube roots, rational, polynomial, exponential, and logarithmic function families algebraically and graphically. Key concepts include
- a) domain, range, and continuity;
- d) zeros;
- e) intercepts; and
- i) vertical and horizontal asymptotes.

Related SOL: AII.7d, f; AII.1c; AII.6

Materials

- Investigating Functions activity sheet (attached)
- What's My Function? activity sheet (attached)
- What's My Function? Graph Stations (attached)
- Graphing utility
- Demonstration tool (document camera, graphing utility, or other)

Vocabulary

asymptote, constant, continuous, dependent variable, decreasing, discontinuous domain, discontinuous range, domain, increasing, independent variable, input, interval, output, point of discontinuity, range, x-intercept, y-intercept, zeros

Student/Teacher Actions: What should students be doing? What should teachers be doing?

Time: 60 minutes

1. Review with students factoring, including factoring out a common factor, the difference of two squares, and a trinomial with a leading coefficient greater than one.
2. Have students graph, with or without a graphing utility, the function $f(x)=\frac{x+2}{x+3}$. Then, have them discuss their graphs with partners and write down any observations they can make about the function. Hold a class discussion, with students sharing their observations.
3. Have students complete, by hand, a table of values for the function, using the trends of the graph to lead them to their chosen domain values. If they are not already doing so after a couple of minutes, have them choose domain values that are smaller than −50 and greater than 50, and values surrounding and including −3. Next, have them see what the value of the function is when *x* is 100 and −100 and when *x* is −2.999 and −3.001.

4. Introduce the term *asymptote*. If you have already taught end behavior and increasing/decreasing intervals, talk about the behaviors surrounding the asymptote in those terms, saying, for example, "A horizontal asymptote of a rational function represents end behavior."
5. Have students graph $f(x)=\frac{x^2-16}{x-4}$ using a graphing utility and then simplify the rational expression on paper. Next, have them graph a second function, $y=x+4$, on their graphing utility without deleting the first function. Discuss why it makes sense that this second graph would be the same line. Then, discuss why it cannot be the same line. Once students have realized that x cannot be 4 in the original function, have them clear the second function from their graphing utility and take a closer look at the rational function. Students may want to use tools within the graphing utility to zoom in and trace the line. By choosing the x-value of 4 or looking at a table, students will discover that there is actually a "hole" in the graph. Have students consider why $x = 4$ is not in the domain of f. (Note: students may need assistance when using graphing utilities.)
6. Introduce discontinuous functions, discontinuous domain, and point of discontinuity, explaining each.
7. Now, have students graph $f(x)=\frac{x-4}{x^2-16}$, factor the denominator and discover which values of x will make the function undefined. Have partners write down everything they notice about this graph. Hopefully, students will discover there is a horizontal asymptote ($y = 0$) as well as a vertical asymptote ($x = -4$). Be sure students realize there is discontinuity where $x = 4$. Have students trace the graph with their finger (or model this on the graph using a demonstration tool such as a document camera or calculator/graphing emulator), exaggerating the point of discontinuity.
8. Have student partners graph the following functions:

   a. $f(x)=\frac{x}{x^2}$ b. $f(x)=\frac{x}{x+1}$ c. $f(x)=\frac{3x}{x+1}$ d. $f(x)=\frac{x^2}{x+1}$

   Direct students to compare the functions algebraically and graphically. Challenge them to make conclusions about horizontal asymptotes based on the algebraic form of the function, as follows:
   - Degree of numerator > degree of denominator ⇒ no horizontal asymptote
   - Degree of numerator < degree of denominator ⇒ horizontal asymptote at $y = 0$
   - Degree of numerator = degree of denominator ⇒ horizontal asymptote at $y = \frac{\text{coefficient of hightest degree term in numerator}}{\text{coefficient of hightest degree term in denominator}}$
9. Have students look back at all of the functions they graphed and determine the x- and y-intercepts. Stress that the zeros of the function are the x-intercepts and that they can often be easily found once a numerator is factored.
10. Distribute copies of the Investigating Functions activity sheet, and have students work in pairs to complete questions 1–6 only without using a graphing utility. As two sets of partners complete the activity sheet, have the four students compare answers and discuss any differences. After each pair has checked and worked through the questions

with another pair, have groups of four join up to compare their work in groups of eight. Finally, have students graph the functions, using a graphing utility to check their answers.

11. Distribute the What's My Function? activity sheet. Have students play a game, competing to see who can write the most functions algebraically. Graphing utilities should not be used! Post the What's My Function? Graph Stations activity sheet on the board, and once students have completed the handout have them write their responses with their initials underneath the corresponding graphs on the board. When they have completed this task, have them graph the various student responses on their graphing utility to determine which functions are correct. The student who wrote the most correct functions wins a prize. (Note: Depending on your class, you may want to make this a partner or larger-group competition.)

Time: 45 minutes

1. Extend the discussion of asymptotes and intercepts to include exponential and logarithmic functions. Now, students should complete questions 7–12 on the Investigating Functions activity sheet. If you have already taught end behavior and domain and range, have students complete the Extension exercise.
2. Place the Functions sheets across the top of the board. For each function, write "*x*-intercepts, *y*-intercepts, horizontal asymptotes, vertical asymptotes," and "points of discontinuity" on separate lines below the function. Give each student an answer card from the Functions activity sheet. (If there are extra cards, some students will receive more than one.) Have students place their cards in the appropriate spaces.

Assessment

- **Questions**
  - What are all asymptotes and points of discontinuity for $f(x)=\dfrac{x^2-5x-14}{2x^2-8}$?
  - What are the *x*- and *y*-intercepts of $f(x)=(-2)^x+3$? Find the answer without using a graphing utility.
- **Journal/writing prompts**
  - Describe how you can determine, without graphing, whether a rational function has any horizontal asymptotes and what the horizontal asymptotes are.
  - Explain how simplifying a rational function can help you determine any vertical asymptotes or points of discontinuity for the function.
- **Other Assessments**
  - Have students film a video of a practical situation involving a quadratic function (e.g., shooting a basketball through a hoop). Have students use editing tools to highlight the presence of a quadratic function. Students will also write a report to capture their selection process, the data collection process, and their conclusions.

  - Research and analyze a country's population growth and/or decay. Students can choose or the teacher can assign a country to research. Students will construct graphs on paper/posters to show the results of their research and analysis.

Extensions and Connections

- Have each student draw his/her own graph with vertical and/or horizontal asymptotes and give the graph to a classmate to write the algebraic function that is graphed.
- Discuss how transformations on the functions would affect location of asymptotes.

Strategies for Differentiation

- Have students play a matching game, either with cards or an interactive whiteboard, to match functions to their graphs.
- Use pictorial representations to reinforce vocabulary.
- Have student find and use a website that demonstrates the relationship between a function and its asymptotes, discontinuity, and intercepts.
- Make extensive use of graphing calculators and/or graphing calculator software to reinforce the characteristics of a rational function.

Note: The following pages are intended for classroom use for students as a visual aid to learning.

Investigating Functions

For each of the following functions, determine the *x*- and *y*- intercepts, discontinuous points, and vertical and horizontal asymptotes.

1. $f(x)=\dfrac{x+2}{x+3}$ *x*- and *y*-intercepts ____________ discontinuous points ____________
   vertical and horizontal asymptotes __________________

2. $f(x)=\dfrac{x^2-9}{x-3}$ *x*- and *y*-intercepts ____________ discontinuous points ____________
   vertical and horizontal asymptotes __________________

3. $f(x)=\dfrac{2x^2-5x-12}{x^2-16}$ *x*- and *y*-intercepts __________discontinuous points ___________
   vertical and horizontal asymptotes __________________

4. $f(x)=\dfrac{5}{x-6}$ *x*- and *y*-intercepts ____________ discontinuous points ____________
   vertical and horizontal asymptotes __________________

5. $f(x)=\dfrac{x+2}{x^2-4}$ *x*- and *y*-intercepts ____________ discontinuous points ____________
   vertical and horizontal asymptotes __________________

6. $f(x) = \frac{1}{x}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

vertical and horizontal asymptotes __________________

7. $f(x) = 2^x$ *x*- and *y*-intercepts ____________ discontinuous points ____________

vertical and horizontal asymptotes __________________

8. $f(x) = 2^{x+2}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

vertical and horizontal asymptotes __________________

9. $f(x) = -(2)^{x-2}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

vertical and horizontal asymptotes __________________

10. $f(x) = \log_2(x)+2$ *x*- and *y*-intercepts ____________ discontinuous points ____________

vertical and horizontal asymptotes __________________

11. $f(x) = \frac{3}{x}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

vertical and horizontal asymptotes __________________

12. $f(x) = \log_3(x)$ *x*- and *y*-intercepts ____________ discontinuous points ____________

vertical and horizontal asymptotes __________________

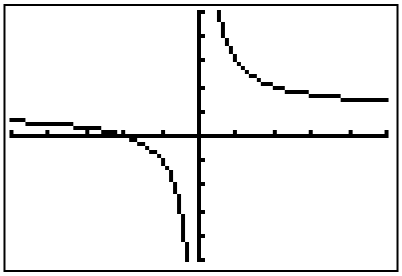
Extension

Using a graphing utility, graph each of the 12 functions above individually. State the domain and range of each function, and describe the end behavior.

What's My Function?

Write the algebraic function for each graph shown.

1.



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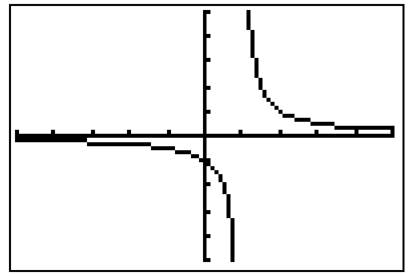
2.



$x = -4$ $y =$

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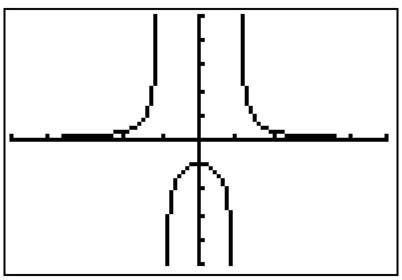
3.



$x = 0$ $y =$

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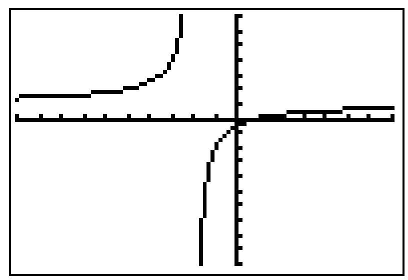
4.



$x = 1$ $y = 0$

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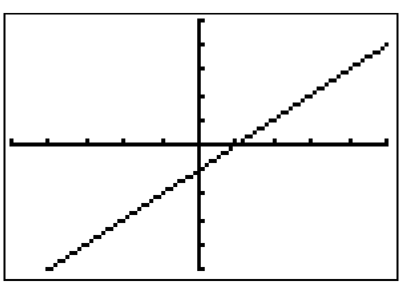
5.



$x = -1$ $y =$

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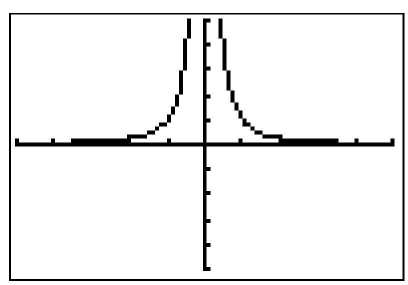
6.



(Challenge)

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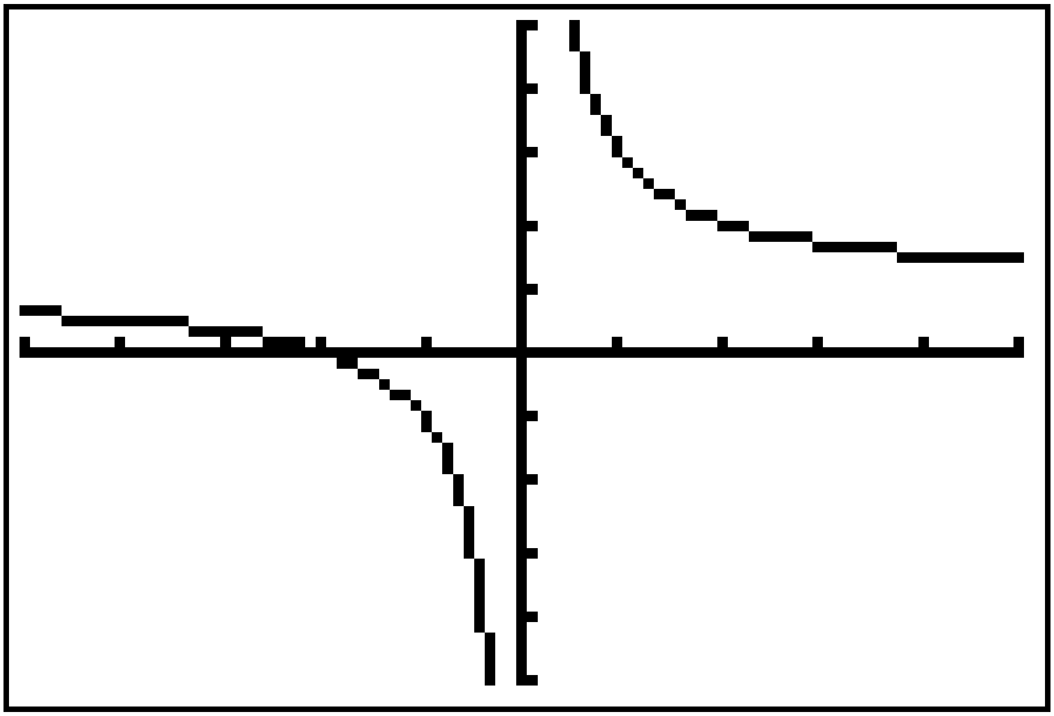
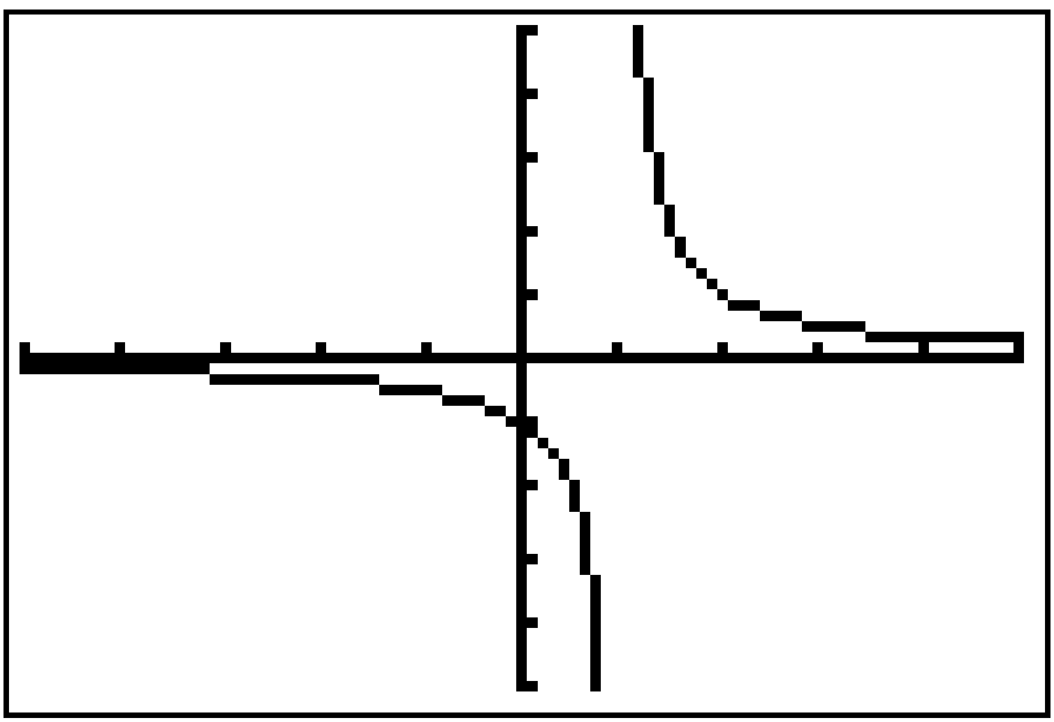
What's My Function? Graph Stations

1.

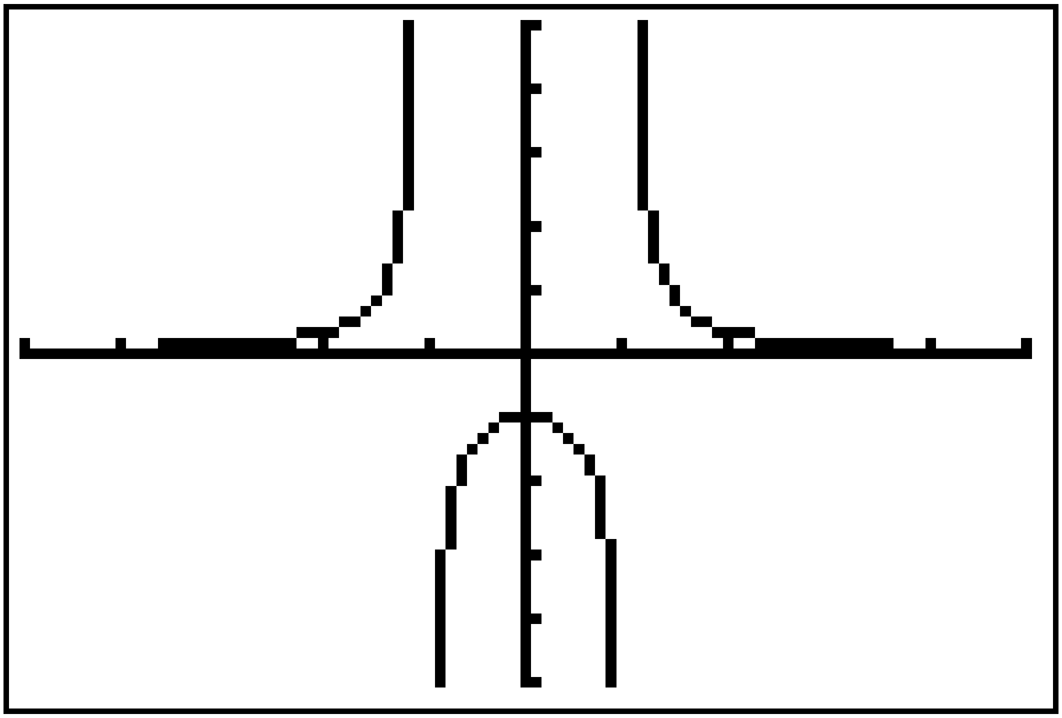


2.



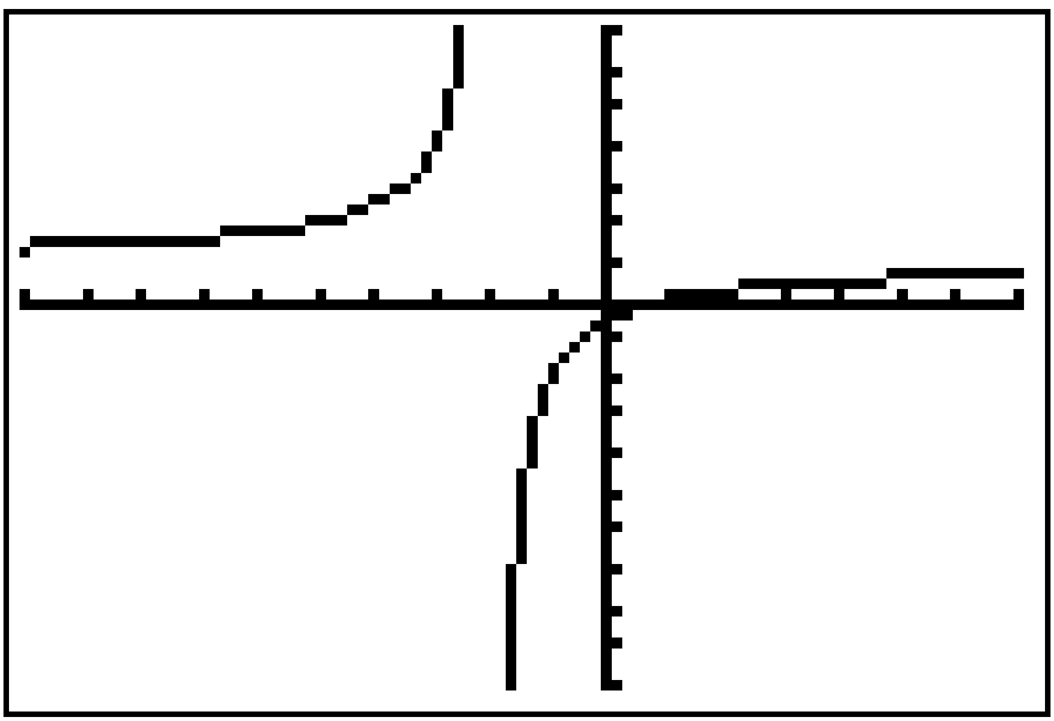
$x = -4$ $y =$

3.



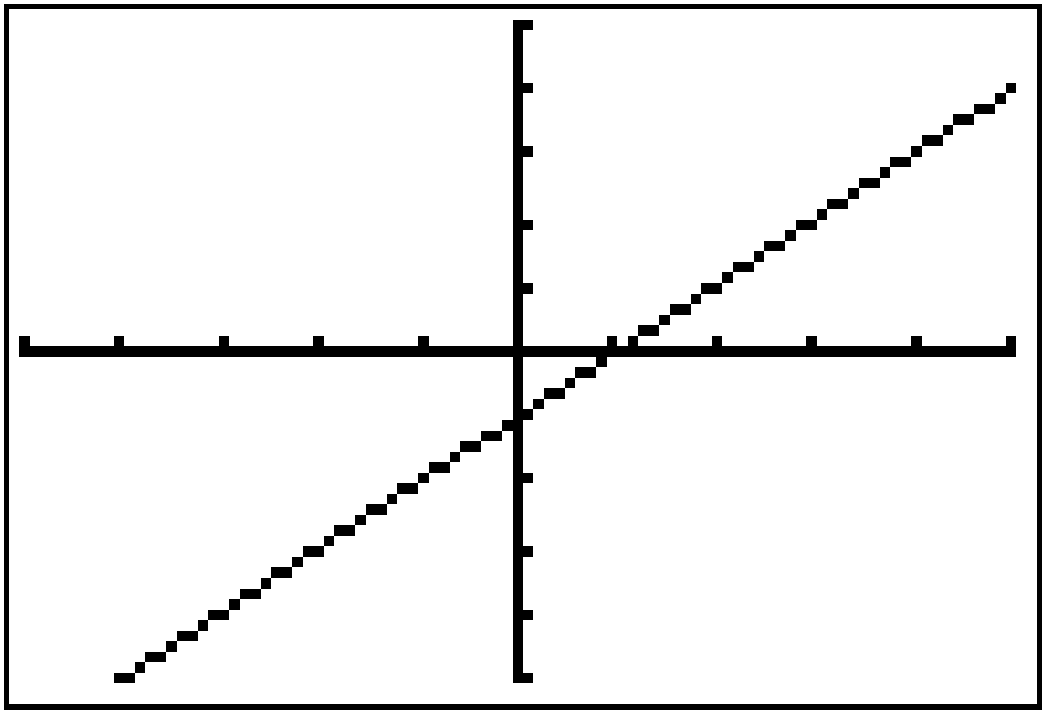
$x = 0$ $y =$

4.



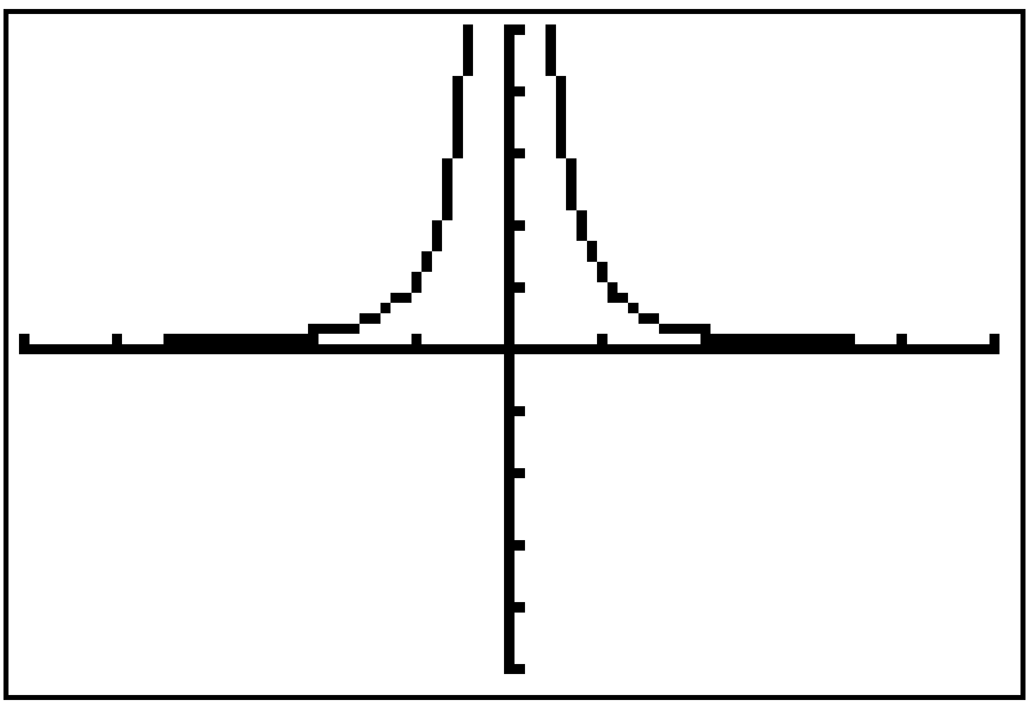
$x = 1$ $y = 0$

5.



$x = -1$ $y =$

6.



(Challenge)

Functions

$$y = \frac{x+3}{x+1}$$

$$y = \frac{x+3}{x^2}$$

$$y = \frac{3x^2 - 5x - 12}{x - 3}$$

$$y = \frac{x^2 - x - 12}{2x^2 - 32}$$

$$5xy = 10$$

$$y = 2^x$$

$$y = 3^x - 3$$

$$y = \log(x)$$

$$y = \log\left(x - 4\right)$$

Answer Cards

Copy cards on cardstock and cut out.

None	$x = 0$
None	$y = 0$

Answer Cards

None	None
$(0,1)$	$y = 0$

Answer Cards

None	$x = -4$
None	$\left(\dfrac{-4}{3}, 0\right)$

Answer Cards

$\left(-3,0\right)$	$\left(0,3\right)$
$x=-1$	$y=1$

Answer Cards

None	$y = 0$
$x = 0$	$(-3,0)$

Answer Cards

None	$(0,4)$
None	None

Answer Cards

Where $x = 3$	Where $x = 4$
None	$y = \frac{1}{2}$

Answer Cards

$\left(-3,0\right)$	$\left(0,\dfrac{3}{8}\right)$
$\left(0,-2\right)$	$\left(1,0\right)$

Answer Cards

$\left(-3,0\right)$	$\left(0,\frac{3}{8}\right)$
$\left(0,-2\right)$	$\left(1,0\right)$

Answer Cards

$(5,0)$	None
$y = -3$	$x = 4$