Arithmetic and Geometric Sequences and Series

Strand: Functions

Topic: Exploring sequences and series

Primary SOL: AII.5 The student will investigate and apply the properties of arithmetic and geometric sequences and series to solve practical problems, including writing the first n terms, finding the nth term, and evaluating summation formulas. Notation will include $\sum$ and a_n.

Materials

- Tic-Tac-Toe activity sheet (attached)
- Applications of Sequences and Series activity sheet (attached)
- Infinite Geometric Sequences activity sheet (attached)
- The Motionless Runner activity sheet (attached)
- Graphing utility

Vocabulary

arithmetic sequence, arithmetic series, common difference, common ratio, explicit formula, geometric sequence, geometric series, infinite series, pattern, recursive formula, sequence, series, summation

Student/Teacher Actions: What should students be doing? What should teachers be doing?

Time: 35 minutes

1. Provide initial instruction on arithmetic and geometric sequences. Review as necessary.
2. Distribute the Tic-Tac-Toe activity sheet. Tell students they will be playing the game after completing the handout. While students are working, write the following numbers on the board in a four-by-three array: $-\frac{63}{16}, -3, -\frac{8}{3}, -2, \frac{3}{5}, 5, 7, 16, 18, 49, \frac{425}{16}, 360$.
3. When students have completed the handout, direct them to work with a shoulder partner to compare their solutions. Students can then check that they have the 12 numbers written on the board as their answers. If any of their answers are different, have them check their work with their partner and/or ask for help.
4. Instruct students to draw a three-by-three tic-tac-toe grid and fill it in with any nine of the 12 numbers written on the board.
5. Randomly call out a problem number from the handout. Have each student look at that problem on his/her sheet, note the answer, and look to see whether the answer is on his/her tic-tac-toe grid. If it is, the student marks out the square.
6. Play continues until a student has three in a row. (Note: If students need additional practice, continue playing until someone has the four corners, a postage stamp, or the entire card filled.)

Time: 60 minutes

1. Divide students into groups of no more than three, and have each group assign a number–1, 2, or 3–to each of its members. Distribute the Applications of Sequences and Series activity sheet. Group members should work together to complete this activity, but each member should individually record all of the work to be turned in. This work must include the formulas and reasons for choosing them, all calculations, and the final answers.
2. Have student groups present and explain the problems to the entire class. Allow each group to choose which problem they want to present, but be sure all six are presented. Instruct groups to allow each member to participate in the presentation.
3. (Optional) Provide students with the formulas for working with arithmetic and geometric sequences and series, but give them neither the explanations of when to use the formulas nor the meanings of the symbols.

Time: 90 minutes

1. Without explaining why, have students look at several examples of infinite geometric series, including common ratios that are greater than one and ratios that are less than one. Have students consider when it is possible to arrive at a sum when a series is infinite.
2. Distribute the Infinite Geometric Sequences activity sheet. Have students complete the sheet, then hold a class discussion about what *infinite* means in this case. Ask: *"Will the ball actually bounce forever?" "What does the sum of the infinite series represent?" "Is an infinite series mathematically possible?"*

Assessment

- **Questions**

  Using the summation notations $\sum_{n-1}^{4} 3n-1$, $\sum_{n-1}^{6} 3(-2)^{n-1}$, and $\sum_{n-1}^{\infty} 3\left(\frac{1}{2}\right)^{n}$, determine

  - the common ratio, r, or the common difference, d
  - the first term, a_1
  - the last term, a_n
  - the sum of the series.

  Check your sums by using the following rules for sums of series:

  arithmetic: $S_n = n\left(\frac{a_1 + a_n}{2}\right)$; geometric: $S_n = \frac{a_1(1-r^n)}{1-r}$; infinite: $S_n = \frac{a_1}{1-r}$

- **Journal/writing prompts**
  - Create a practical situation in which a geometric series would be used.
  - Compare the use of the rules for arithmetic series and arithmetic sequences with summation notation (∑) for an arithmetic series.
  - Explain what the common ratio *cannot* be if we are to calculate an infinite series, and explain why the value of the common ratio is restricted for an infinite geometric series.

- **Other Assessments**

  - Have students discuss in pairs common characteristics of geometric series in practical situations.
  - Create an exit slip to have students evaluate an example summation.

Extensions and Connections

- Have students solve the following problems individually and explain completely how they arrived at their solutions. They may use words only, formulas with explanations, and/or clear diagrams with explanations.
  - A person just fitted for contact lenses is told to wear them only two hours the first day and to increase the length of time by 20 minutes each day. After how many days will the person be able to wear the contacts for 14 hours?
  - A wooden ladder is wider at the bottom than at the top and has 10 equally spaced rungs (steps). The bottom rung is 22 inches long and the top rung is 14 inches long. What is the total length in inches of all the rungs together?
  - A post is driven into the ground. The first strike drives the post 30 inches into the ground. The next strike drives the post 27 more inches into the ground. Assume these distances form a geometric sequence. What is the total distance (to the nearest inch) the post is driven into the ground after eight strikes? What is the maximum distance the post could be driven?
  - Poor Mark! His dog ate his math homework. He remembered that the first three terms of his arithmetic sequence were 4, 8, and 12, and the last two were 356 and 360. How many terms were there in this sequence?
- Lead students in graphing an arithmetic series and a geometric series. Discuss the linear relationship of the terms of an arithmetic series, and introduce exponential functions while discussing the graph of the geometric series. Compare the graphs of geometric series for $r > 1$ and for $r < 1$.
- Have students complete The Motionless Runner activity sheet. Then have students discuss the activity in small groups. Have groups report out as part of a class discussion of the problem.

Strategies for Differentiation

- Provide multiple-choice answer options for the Applications of Sequence and Series activity sheet, as well as the Activity Extension and Connections problems above.
- Allow students to use talking graphing utility software.
- Use vocabulary cards for related vocabulary listed above.

Note: The following pages are intended for classroom use for students as a visual aid to learning.

Tic-Tac-Toe

Sequences and Series

1. If $a_1 = 24$ and $a_3 = -8$ determine a_2

2. Find a_{10} for the arithmetic sequence $-8, -7\frac{1}{3}, -6\frac{2}{3}, \ldots$

3. Find a_1 if $a_8 = 23$ and $d = \frac{16}{5}$.

4. How many terms are in the series 5 + 7 + 9 + … + 99 + 101?

5. Find the 20th term of $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \ldots$

6. Find the sum of $\sum_{n=1}^{15} 3k$.

7. If $a_1 = 8$ and $a_5 = 40.5$, determine a_2, a_3, a_4

8. What is the common ratio of the series modeled by $\sum_{n=1}^{6} 4(-3)^{n-1}$?

9. Using $S_n = \dfrac{a_1(1-r^n)}{1-r}$, find the sum of the geometric series $-6, 3, -\dfrac{3}{2}, \ldots, \dfrac{3}{16}$.

10. How many terms are in the geometric sequence having a first term 2, a last term 32, and a common ratio −2?

11. Evaluate $\sum_{n=1}^{4} 20\left(\dfrac{1}{4}\right)^{n-1}$.

12. How many terms are in a sequence if the last term, a_7, is 14?

Applications of Sequences and Series

1. A runner begins training by running 5 miles one week. The second week she runs a total of 6.5 miles. The third week she runs 8 miles. Assume this pattern continues.
   - How far will she run in the 10th week?
   - At the end of the 10th week, what will be the total distance she has run since she started training?
   - Express the total distance with summation notation (∑).

2. A superball is dropped from a height of 2 meters and bounces 90 percent of its original height on each bounce.
   - When the ball hits the ground for the eighth time, how far has it traveled?
   - How high off the floor is the ball at the top of the eighth bounce?

3. A snail is crawling straight up a wall. The first hour it climbs 16 inches, the second hour it climbs 12 inches, and each succeeding hour, it climbs only three-fourths the distance it climbed the previous hour. Assume the pattern continues.
   - How far does the snail climb during the seventh hour?
   - What is the total distance the snail has climbed in seven hours?
   - Express the total distance with summation notation ($\sum$).

4. Suppose on January 1 you deposit $1 in an empty piggy bank. On January 8, you deposit $1.50; on January 15, you deposit $2; and each week thereafter you deposit 50 cents more than the previous week.
   - What kind of sequence do these deposits generate?
   - What amount will you deposit in the 52nd week?
   - What is the total in the piggy bank at the end of these 52 weeks?

5. Carla's Clothing Shop opened eight years ago. The first year, she made $3,000 profit. Each year thereafter, her profits averaged 50 percent greater than the previous year.
   - How much profit did Carla earn during her 18th year of business?
   - What was the total amount of profit Carla earned over her first 18 years?

6. A ball on a pendulum moves 50 centimeters on its first swing. Each succeeding swing it moves 0.9 centimeters more than the previous swing.
   - Write the first six terms of the sequence generated.
   - Assuming the pattern continues, how far will the ball travel before coming to rest?

7. Find the sum of the odd integers from 25 to 75, inclusive.

Infinite Geometric Sequences

On each bounce, Ball 1 reaches a height equal to three-fourths the height of its previous bounce. On the first bounce, it achieves a height of 25 feet.

Ball 2, which reaches a height of 18 feet on its first bounce, bounces four-fifths the height of its previous bounce on each bounce.

1. Complete the table, showing the height of each ball for each bounce.

Bounce	Ball 1 height (ft.)	Ball 2 height (ft.)
1		
2		
3		
4		
5		
6		
7		

2. Will Ball 2 ever bounce higher than Ball 1? If so, at which bounce?

3. For how many bounces do both balls bounce above 10 feet?

4. When do the balls "stop" bouncing (i.e., achieve a height of less than 3 inches)?

5. What minimum initial bounce height (to the nearest foot) would you have to ensure for each ball in order to guarantee that it bounces at least 8 feet high by the sixth bounce?

6. Calculate the sum of all of the bounce heights for both balls.

Follow-up

7. What is it about the fractions involved in this problem that makes the ball react in the manner described?

The Motionless Runner

A runner wants to run a certain distance—say, 100 meters—in a finite length of time. To reach the 100-meter mark, the runner must first reach the 50-meter mark, and to reach that, he must first run 25 meters, but to do that, he must first run 12.5 meters. Since space is infinitely divisible, we can repeat these “requirements” forever, or infinitely. Thus, the runner has to reach an infinite number of midpoints in a finite time. Doing this is impossible, of course, so the runner can never reach his goal. In other words, in general, anyone who wants to move from one point to another must meet these requirements, and so *motion is impossible*. What we perceive as motion is merely an illusion.

1. Where does this argument break down?

2. Why?