Factoring

Strand: Expressions and Operations

Topic: Factoring Polynomials

Primary SOL: AII.1 The student will

c) factor polynomials completely in one or two variables.

Related SOL: AII.8

Materials

- Finding the Greatest Common Factor activity sheet (attached)
- Factoring the Greatest Common Factor activity sheet (attached)
- Factoring by Grouping activity sheet (attached)
- Factoring Trinomials $ax^2 + bx + c$ activity sheet (attached)
- Factor Pattern Examples activity sheet (attached)
- Factoring Summary activity sheet (attached)
- Factor Patterns activity sheet (attached)
- Exit Ticket Questions (attached)
- Graphing utility

Vocabulary

binomial, common binomial factor, complete factorization, difference of two cubes, difference of two squares, distributive property, factor, factor theorem, greatest common factor (GCF), perfect square trinomial, polynomial, prime, substitution property, sum of two cubes, trinomial

Student/Teacher Actions: What should students be doing? What should teachers be doing?

Time: 60 minutes

<u>Greatest Common Factor (GCF)</u>

If students cannot find *the GCF of algebraic expressions by inspection, start here.*

1. Review finding the prime factors of two or more composite numbers (e.g., 12 can be factored as $4 \cdot 3$, $6 \cdot 2$, or $2 \cdot 2 \cdot 3$). Determine the product of their shared prime factors, and call it the GCF.
2. Use the same strategy in Step 1 to find the GCF of two or more composite expressions. Have students complete the Finding the Greatest Common Factor activity sheet.

If students can find the GCF of algebraic expressions by inspection, start here.

3. Demonstrate how to factor the GCF of polynomial expressions. Include expressions where the terms are relatively prime to one another.
4. Show the students how to apply the distributive property when checking the accuracy of factors. Have students work in pairs and complete the Factoring the Greatest Common Factor activity sheet.

Time: 30 minutes

Factoring by Grouping

5. Using the Factoring by Grouping activity sheet, demonstrate how to use the GCF to factor polynomials by grouping. Explain to students that the GCF can also be a binomial, hence the term *common binomial factor*. Apply the distributive property to check the accuracy of factors.

Time: 90 minutes

Factoring Trinomials of the Form $ax^2 + bx + c$

6. Before starting any other method of factoring, remind students to always check for the GCF and factor it out first.
7. Have students multiply the binomials $(2x + 5)$ and $(3x - 4)$. Explain that the result is a polynomial with four terms but can become a trinomial by the substitution property, where $-8x + 15x = 7x$.

$$(2x + 5)(3x - 4) = 6x^2 - 8x + 15x - 20 = 6x^2 + 7x - 20$$

8. Demonstrate how to change a trinomial $ax^2 + bx + c$ into a polynomial with four terms by finding the factors of ac whose sum is b. Then, use factoring by grouping to find the binomial factors. Remind them that the focus is identifying two numbers with a given product, *c*, and a given sum, *b*.

    - Factoring $ax^2 + bx + c \quad when \;\; a = 1$

$x^2 - x - 30 = x^2 - 6x + 5x - 30$ — *factors of -30 (ac) whose sum is – 1 (b) are – 6 and 5*

$x^2 - x - 30 = (x^2 - 6x) + (5x - 30)$ — *group the terms*

$x^2 - x - 30 = x(x - 6) + 5(x - 6)$ — *factor the GCG from each group*

$x^2 - x - 30 = (x - 6)(x + 5)$ — *factor out the common binomial*

    - Factoring $ax^2 + bx + c \quad when \;\; a > 1$

$6x^2 - 7x - 20 = 6x^2 + 8x - 15x - 20$ — *6(– 20) = – 120 Factors of – 120 whose sum is – 7 are 8 and – 15*

$6x^2 - 7x - 20 = (6x^2 + 8x) + (-15x - 20)$ **group the terms*

$6x^2 - 7x - 20 = 2x(3x + 4) - 5(3x + 4)$ — *factor out the GCF from each group*

$6x^2 - 7x - 20 = (3x + 4)(2x - 5)$ — *factor out the common binomial*

*Emphasize the common error when converting a polynomial into a binomial by grouping.

9. Have students complete the Factoring Trinomials $x^2 + bx + c$ activity sheet. When students are permitted to work in pairs, one student should complete the first column

while the other student completes the second column. Then, have students exchange their work to check for accuracy.

10. Students who have difficulty with traditional grouping may be able to visual the process by relating it back to the box method used for multiplying. Students start the process the same way they would do traditional grouping but use the boxes to group the terms instead.

    Example:

    $6x^2 - 7x - 20$

	$6x^2$	
		-20

    $6x^2 + 8x - 15x - 20$

	$6x^2$	$8x$
	$-15x$	-20

    Have students look for the GCF in each column and row. Students could also look for the GCF in the first row or column and then decide what should go in the next spot to make the appropriate products.

$2x$	$6x^2$	$8x$
	$-15x$	-20

→

	$3x$	
$2x$	$6x^2$	$8x$
	$-15x$	-20

→

	$3x$	4
$2x$	$6x^2$	$8x$
-5	$-15x$	-20

    Factored form: $(3x+4)(2x-5)$

Time: 90 minutes

<u>Factoring Patterns</u>

11. Through examples of multiplication of binomials, lead students to discover the following factor patterns:
    - Difference of Two Squares $a^2 - b^2 = (a-b)(a+b)$
    - Perfect Square Trinomial $a^2 + 2ab + b^2 = (a+b)(a+b)$
      $a^2 - 2ab + b^2 = (a-b)(a-b)$
    - Sum of Two Cubes $a^2 + b^2 = (a+b)(a^2 - ab + b^2)$
    - Difference of Two Cubes $a^2 - b^2 = (a-b)(a^2 + ab + b^2$
12. Show examples of how to factor a polynomial using the factoring patterns. Let students work on the exercises in Activity Sheet 5: Factor Pattern Examples.

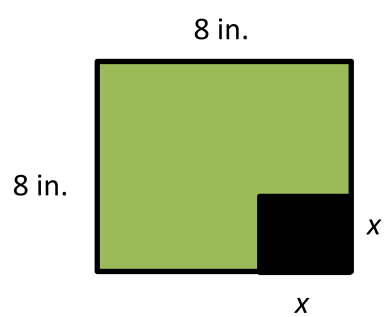
13. For additional, practice distribute and have students complete the Factor Patterns activity sheet.

Summary

14. Have students work in pairs to complete the graphic organizer in the Factoring Summary activity sheet, showing all of the methods of factoring polynomials.
15. Have students classify the polynomials on Activity Sheet 6: Factoring Summary handout by factor pattern and have them factor the polynomials completely.
16. Create an exit ticket for students using the provided assessment questions (attached).

Assessment

- **Questions**
  - Is it possible to factor the sum of two squares? Justify your answer by showing and/or explaining an example.
  - What are the four special factor patterns?
  - How do you know that a polynomial expression is in a completely factored form?
  - Suppose a square of side *x* is cut from an 8” x 8” piece of cardboard. Express the area of the remaining piece as a polynomial expression and as a polynomial in factored form.



- **Journal/Writing Prompts**
  - Explain why $w^2 + 25$ is not considered as a perfect-square binomial.
  - Write a song to help remember the steps taken to completely factor a polynomial expression.
  - A student factored the expression below. If the answer is right, show mathematically that the factors are correct. If the answer is wrong, find the error and factor the expression correctly.

$$3x^2 - 8x + 5$$

$$3x^2 - 5x - 3x + 5$$

$$(3x^2 - 5x) + (3x - 5)$$

$$x(3x - 5) + 1(3x - 5)$$

$$(3x - 5)(x + 1)$$

Extensions and Connections (for all students)

- Have students graph the following functions, one at a time, using a graphing utility:

  $y = x^2 - 2x - 15$ $y = x^2 + 6x + 9$ $y = x^2 - 4$ $y = 4x^2 + 12x + 5$

  Then, have students factor the polynomial expressions. After completing the four examples, ask, *"How are factors and x-intercepts related?"* Generalize students' conjecture by introducing the factor theorem.

  $$Factor\ Theorem:$$
  $$The\ binomial\ \ (x-a)\ is\ a\ factor\ of\ the\ polynomial\ f(x)\ \ if\ and\ only\ if\ f(a)=0$$

- Give students several expressions in the form $ax^2 + bx + c$, and have them evaluate $b^2 - 4ac$ and graph $f(x) = ax^2 + bx + c$ using a graphing utility. Have students complete the table.

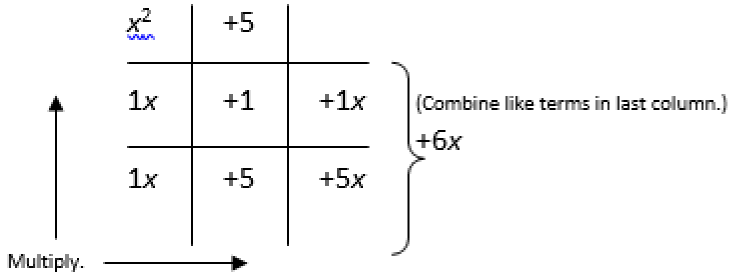
$ax^2 + bx + c$	$b^2 - 4ac$	How many x-intercept(s) does the graph have?
$x^2 - 4x + 4$		
$x^2 + 6x + 8$		
$x^2 + 4$		

- Based on the answers in the table, ask students, *"How can the value of the discriminant predict the number of x-intercepts of the graph?"*

Strategies for Differentiation

- For advanced classes, consider using the activity sheets 2 through 7 as learning stations, and have a debriefing at the end of the activity.
- Another variation to the lesson is to go to the graphic summary in the Factor Pattern Examples activity sheet, then post a copy of activity sheets: 2 through 5 and 7 in posters around the room. Put the students in heterogeneous groups of three to four. Give each student a marker and let them do a walk about from each poster. As a group, they are expected to choose two questions from each poster, discuss how to factor the expression, and write their final answer on the poster.
- Use algebra tiles or geometric figures to model factoring trinomials, perfect-square trinomials, and the difference of two squares.
- Show a YouTube video showing the geometric interpretation of the sum and difference of two cubes and its factors.
- Create a matching game that uses cards showing expressions in expanded form and corresponding cards showing the expressions in factored form.
- Have students use a graphing utility to find the factors of a number. For example, to find the factors of 36, have them graph $y = \frac{36}{x}$ and use the table to find integral (x, y) pairs.

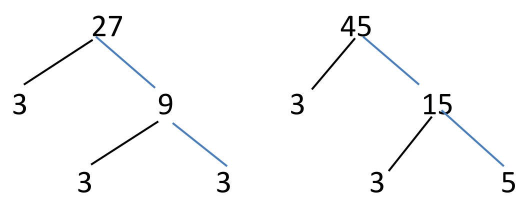
- Use a tic-tac-toe grid to illustrate factoring a trinomial. For example, the grid below illustrates a method for factoring $x^2 + 6x + 5$.



- The slip-and-slide method of factoring a trinomial can be introduced as an alternative method. For a conceptual justification of this method, refer to the article "A Transformational Approach to Slip-Slide Factoring," by Jeffrey Steckroth. *The Mathematics Teacher, (109)* 3, October 2015, 228-234.
- Have students use an acronym to remember the sign of the factored form of the sum or difference of two cubes.
- Use the factoring handouts in this lesson to create a game, such as a tic-tac-toe game or a matching game.

Note: The following pages are intended for classroom use for students as a visual aid to learning.

Finding the Greatest Common Factor

<table>
<tr>
<td>Find the prime factors using the factor tree method

The shared common factor is the GCF
GCF = 3 (3) = 9</td>
<td>Find the prime factors using the table method
$27x^2y^5 = 3 \cdot 3 \cdot 3 \cdot x^2 \cdot y^3$
$45x^3y^2 = 3 \cdot 3 \cdot 5 \cdot x^3 \cdot y^2$
GCF = $3 \cdot 3 \cdot x^2 \cdot y^2$
= $9x^2y^2$
• the common variable is the one with the lower exponent</td>
</tr>
<tr>
<td colspan="2">Start at No. 1 and find the greatest common factor of the expression. The answer should be found in another box, which will be called No. 2. Then find the greatest common factor of the expression in that box and find the answer in another box. If you can complete the whole activity, your answers are all correct.</td>
</tr>
<tr>
<td>#1   9
$4m^3 - 7m^2$</td>
<td>#_____ m
$3m^2n - 3m^2n^6$</td>
</tr>
<tr>
<td>#______ $3m$
$2m^4n^2 - 4m^3n^5 + 6m^2n^3$</td>
<td>#_____ $4mn$
$9m^2 - 27mn + 63n^2$</td>
</tr>
<tr>
<td>#______ $5m$
$8m^2n + 24mn^2 - 20mn$</td>
<td>#___ m^2
$9m - 63mn + m^2$</td>
</tr>
<tr>
<td>#_____ $3m^2n$
$-9m^5 + 18m^2 - 3m$</td>
<td>#_____ $2m^2n^2$
$35m^2 - 15m$</td>
</tr>
</table>

Factoring the Greatest Common Factor

Factor out the GCF in each polynomial expression

1) The greatest common factor of the expression is ________________. 2) Divide each term by the greatest common factor. 3) Write the final answer.	$27x^2y^5 + 45x^3y^2 + 9x^2y^2$ $\frac{27x^2y^5}{9x^2y^2} + \frac{45x^3y^2}{9x^2y^2} + \frac{9x^2y^2}{9x^2y^2}$ $9x^2y^2\ (3y^3 + 5x + 1)$
Polynomials $27x^2y^5 + 45x^3y^2 + 9x^2y^2$	**Polynomials in Factored Form** $9x^2y^2\ (3y^3 + 5x + 1)$
$m^2 - 4m$	
$20m^2n - 15mn^2$	
$-12mn + 16$	
$5x^2y - 3xy + 15y^2$	
$45x^2y + 15xy^2 - 3xy$	
$9y^3 + 18y^2 - 6y$	
$8ab^2 - 12ab + 4$	
$7a^2b - 28ab - ab^2$	
$9 - 12ab + 15b^2$	

Factoring by Grouping

Factor the given polynomial. 1) Group two terms with a common factor 2) Factor the GCF from each group 3) Factor the Greatest Common Binomial Factor	$8x^2 + 4xy - 10xy^2 - 5y^3$ $(8x^2 + 4xy) + (-10xy^2 - 5y^3)$ $4x(2x + y) - 5y^2(2x + y)$ $(2x + y)(4x - 5y^2)$
$9x^2 + 6xy^2 + 12x + 8y^2$	$4x^2 - 5xy^2 - 16x + 20y^2$
$3x^2 - x - 15x + 5$	$x^2 - 8x - 9x + 72$
$8ab + 28b - 10a^2 - 35a$	$2a^2 - 18ab + 7a - 63b$
$m^3 - m^2 + 10m - 10$	$m^3 - 7m^2 + m - 7$
$y^2 + 5y - y - 5$	$6y^2 - 3y + 4y - 2$

Factoring Trinomials $ax^2 + bx + c$

Factor each polynomial completely. If a greatest common factor other than 1 exists, factor it first.

$x^2 - x - 72$	$x^2 + 2x - 63$
$x^2 + 15x + 50$	$x^2 - 15x + 26$
$x^2 + 12xy + 36y^2$	$x^2 - 2xy + y^2$
$3y^2 - 24xy - 60x^2$	$4x^2 - 20xy + 24y^2$
$2a^2 + 7a - 30$	$6a^2 - a - 1$
$15b^2 - 8b + 1$	$8b^2 + 22b + 5$
$3a^2 - 16ab + 5b^2$	$4b^2 + 7ab + 3a^2$
$8x^2 + 10x - 12$	$9x^2 - 48x - 36$

Factor Pattern Examples

Factor the following polynomials completely. If the polynomial is nonfactorable, write prime.

A. DIFFERERENCE OF SQUARES $a^2 - b^2 = (a+b)(a-b)$

$(x+1)^2 - 121$	$4x^2 - 49y^2$
$a^2 + 81$	$2x^2y^2 - 200$

B. PERFECT-SQUARE TRINOMIALS $a^2 \pm 2ab + b^2 = (a \pm b)(a \pm b)$

$m^2 - 10m + 25$	$4f^2 + 12f + 9$
$49x^2 + 28xy + 4y^2$	$3p^2 - 30p + 75$

C. SUM AND DIFFERENCE OF CUBES $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

$c^3 - 8$	$2y^3 + 128$
$5 + 5n^3$	$4m^4 - 108m$

Factoring Summary

FACTOR OUT THE GCF, if it exists

BINOMIALS

POLYNOMIALS 4 TERM

TRINOMIALS

DIFFERENCE OF SQUARES

$$\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$$

$25x^2 - 64y^2 =$

SUM OF CUBES

$$\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$

$125x^4 + x =$

DIFFERENCE OF CUBES

$$\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$$

$8x^3 - 27y^3 =$

PERFECT-SQUARE TRINOMIALS

$$\boldsymbol{a^2 \pm 2ab + b^2 = (a \pm b)(a \pm b)}$$

$9x^2 - 12x + 4 =$

TRINOMIALS (a = 1)

$x^2 - 7xy - 30y^2 =$

TRINOMIALS (a > 1)

$5x^2 + 11x + 2 =$

GROUPING

$x^2y + xy - 4x - 4 =$

Factor Patterns

Classify the following polynomials according to factor patterns and copy the polynomial in the appropriate box. Then, factor each polynomial completely.

1. $m^2 + 2m - 15$
2. $x^2 - 16$
3. $9a^2 + 6a - 8$
4. $v^3 + 1$
5. $4w^2 + 12w + 9$
6. $k^2 + 2km + m^2$
7. $128 - 2x^2$
8. $3q^2 + 3q + 12$
9. $8h^2 - 8h - 6$
10. $9r^3 - 27$
11. $225n^4 - p^6$
12. $16c^2 + 40cd + 25d^2$
13. $4y^2 + 4$
14. $a^6 - b^6$

Trinomial	Perfect Square Trinomial
Difference of Two Cubes	**Sum of Two Cubes**
Difference of Two Squares	**Greatest Common Factor**

Exit Ticket Questions

1. Which binomial is a factor of $4x^2 - 13x + 3$?

   A $2x - 3$
   B $2x - 1$
   C $4x - 3$
   D $4x - 1$

2. Choose all factors for $x^2 - 5x - 36$.

x + 4	x + 9	x + 3	x – 9
x – 18	x – 12	x + 2	x – 4

3. Identify all factors of the expression $4x^3 - 9xy^2$.

(xy)	(2x – 3y)	(4x + 9y)	(x)
($4x^2$ – 9)	(xy^2)	(2x+3y)	(4x – 9y)

4. Choose all factors for $5x^2 + 4xy - 12y^2$.

x + 2y	x – 2y	x + 10y	5x + 10y
5x – 6y	x – 6y	5x – 12y	5x + 6y

5. Which of the following is a true mathematical statement?

   A $8x^3 - 27y^3 = (2x - 3y)(4x^2 + 12xy + 9y^2)$
   B $8x^3 - 27y^3 = (2x - 3y)(4x^2 - 6xy + 9y^2)$
   C $8x^3 - 27y^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$
   D $8x^3 - 27y^3 = (2x - 3y)(4x^2 - 12xy + 9y^2)$
   E $8x^3 - 27y^3 = (2x - 3y)(2x - 3y)(2x - 3y)$

6. Given the polynomial $(m + 5)^2 - 9$, which of the following is equivalent to the above expression?

    a) $(m + 5 + 3)(m + 5 - 3)$
    b) $(m - 5 + 3)(m - 5 - 3)$
    c) $(m + 5 + 9)(m + 5 - 9)$
    d) $(m - 5 + 9)(m + 5 - 9)$

7. Factor the expression: $4y^3 + 32x^3$.

8. Factor the expression: $3x^2 + 2xy + 10y + 15x$.

9. Factor the following expression: $x^4 - 1$.

10. Find all factors of the following expression: $5m^2n^2 - 10mn^2 - 15n^3$.