Inverse Functions

Strand: Functions

Topic: Exploring Inverse Functions

Primary SOL: AII.7 The student will investigate and analyze linear, quadratic, absolute value, square root, cube root, rational, polynomial, exponential, and logarithmic function families algebraically and graphically. Key concepts include

j) inverse of a function.

Related SOL: AII.7a, g, k

Materials

- Exploring Inverse Functions activity sheet (attached)
- Finding the Inverse of a Function activity sheet (attached)
- Graphing utility

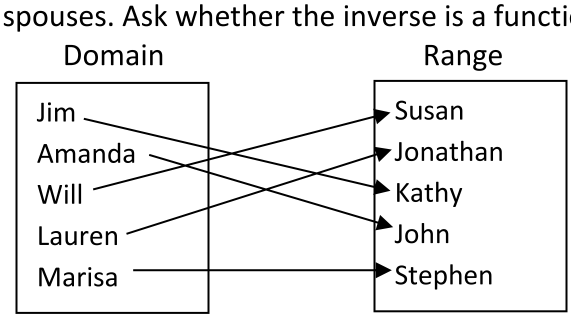
Vocabulary

dependent variable, domain, independent variable, input, one-to-one function, output, range, reflection, vertical line test, line of reflection

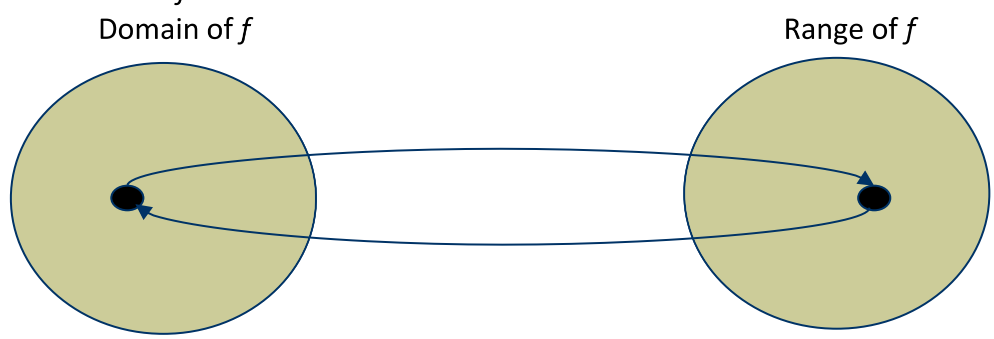
Student/Teacher Actions: What should students be doing? What should teachers be doing?

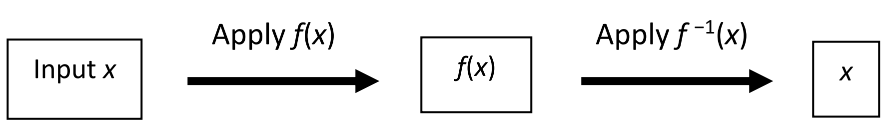
Time: 90 minutes

1. Let students work in groups and complete the Exploring Inverse Functions activity sheet as a warm-up activity. Choose a completed paper from each group and post it around the room. Have students do a quick gallery walk to see the output from each group. Debrief the activity by discussing students' responses to these guiding questions for the activity.
   - How do you find the area of the square, given the length of sides?
   - How do you find the length of sides, given the area?
   - How are the graphs of the two functions related with respect to the line $= x$?
2. Transition into defining the inverse of a function as a set of ordered pairs in which $(a,b) \in f$, then $(b,a) \in$ inverse of f .
3. Using the definition above, have students find the inverse of the function below, where the elements of the domain are friends of Monica, and the range consists of the friends' spouses. Ask whether the inverse is a function.

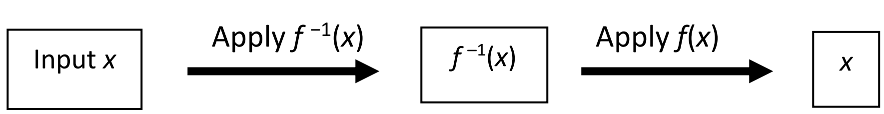


4. Display the function {(−3, 27), (−2, −8), (−1, −1), (0, 0), (1, 1), (2, 8), (3, 27)}, and have students find the inverse. Ask whether the inverse is a function. Then, have students find the inverse of the function {(−3, 9), (−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4), (3, 9)}, and again ask whether the inverse is a function.
5. Ask students how they can tell by looking at a function if its inverse will also be a function. Lead them to see that functions in which the unique inputs (values of x from the domain) correspond or “map” to unique outputs (values of $f(x)$ from the range) have an inverse that is a function. Tell them that this is referred to a *one-to-one function.*
6. Discuss the vertical line test—what it is and why it works. Ask whether a horizontal line test would tell us anything useful. Have students graph $y = -0.5(x + 2)^2 + 3$ and then graph the horizontal line $y = 2$. Ask, *“Where does the line intersect the parabola?” “Why does the horizontal line test work?”*
7. Explain that a function $y = f(x)$ is a rule that tells us to do something with the input, x. Hence, $f(x) = 2x$ tells us to multiply the input by 2. Emphasize that the inverse of f undoes” whatever f does; therefore, $f^{-1}(x) = \frac{1}{2}x$, because $\frac{1}{2}$ is the multiplicative inverse of 2. Show students that to find the inverse of $f(x) = 3x - 1$, you perform the inverse operations in reverse order. That is, begin with your input, x, add 1, and then divide by 3: $f^{-1} = \frac{x+1}{3}$.
8. Demonstrate that for a function $y = f(x)$ to have an inverse function, f must be *one-to-one*: that is, for every x in the domain, there is exactly one y in its range, and likewise, each y in the range corresponds to exactly one x in the domain. The correspondence from the range of f onto the domain of f is, therefore, also a function. This function is the inverse of f.





$f^{-1}[f(x)] = x$



$f[f^{-1}(x)] = x$

9. Ask students to determine whether each pair of functions below is an inverse of one another.

   $h(x) = x^3$ and $g(x) = \sqrt[3]{x}$

   $h[g(x)] = x$ and $g[h(x)] = x$

   $h(\sqrt[3]{x}) = x$ and $g(x^3) = x$

   $\left(\sqrt[3]{x}\right)^3 = x$ and $\sqrt[3]{x^3} = x$

   Students should find that they are indeed inverses. Tell them that because the pairs are inverses of one another, the composition of the two functions is equal to x; that is, f(g(x)) = g(f(x)) = x.

10. Have students enter $Y = x^3$ and $Y = \sqrt[3]{x}$ in their graphing utility and adjust the settings for the graph view so that $-3 \le x \le 3$ and $-2 \le y \le 2$. Ask what they observe about the graph. Then, have them enter $Y_3 = x$. Have students note the symmetry of the inverse functions with respect to $y = x$. Tell them to think about reflections. Use functions of the graphing utility, such as a table, point out that if (a, b) is on the graph of *f,* then (b, a) is on the graph of f^{-1}. (Note: students may need assistance with the functionalities of different graphing utilities as they explore inverse functions in this lesson.)

11. Finding the inverse: Inform students that the graph of a one-to-one function, *f*, and its inverse are symmetric with respect to $y = x$. Therefore, we can identify f^{-1} by interchanging the roles of *x* and *y*: if *f* is defined by the equation $y = f(x)$, then f^{-1} is defined by the equation $x = f(y)$. The equation $x = f(y)$ defines f^{-1} implicitly. Solving for *y* will produce the explicit form of f^{-1} as $y = f^{-1}(x)$.

12. Give students the function $f(x) = 2x + 3$, and ask whether *f* is one-to-one. (Yes, it is linear and increasing.)

    $f(x) = 2x + 3$

    $y = 2x + 3$

    $x = 2y + 3$

    The variables *x* and *y* have been interchanged in the original equation. This equation defines f^{-1} implicitly. Solving for *y*,

    $2y + 3 = x$

    $2y = x - 3$

    $y = \frac{1}{2}(x - 3)$

    $f^{-1}(x) = \frac{1}{2}(x - 3)$ This is the explicit form.

    What is the domain of *f* ? What is the domain of f^{-1}?

    What is the range of *f* ? What is the range of f^{-1}?

13. Have students graph $Y_1 = f(x) = 2x + 3$ and its inverse, $Y_2 = f^{-1}(x) = \frac{1}{2}(x - 3)$, with $Y_3 = x$. Point out the symmetry of the graphs with respect to $y = x$.
14. Show several examples of finding the inverse both algebraically and graphically using other functions and discuss how to find the domain and range of the function and its inverse. Then determine whether the inverse is a function or not.
    $f(x) = (x-3)^2 + 4,\ x \geq -2$
    $f(x) = \sqrt[3]{x-4} + 2$
    $f(x) = \frac{2x+3}{x-4}$
15. Have students complete the Activity Sheet 2: Finding the Inverse of a Function handout. The activity focuses on finding the inverse both algebraically and graphically.

Assessment

- **Questions**
    - On graph paper, graph the functions $f(x) = \frac{3}{4}x + 2$ and $g(x) = \frac{4}{3}x - 2$. Are these functions inverses of one another? Why or why not?
    - Graph $f(x) = -3x + 5$. Now, graph $y = x$. Without manipulating the original function algebraically, graph $f^{-1}(x)$. Use the line of reflection to determine points on the inverse function.
- **Journal/writing prompts**
    - Describe, in detail, three methods for finding the inverse of $f(x) = \frac{2}{3}x - 4$.
    - Explain what it means for a function to be "one-to-one," and describe two methods for determining whether or not a function is one-to-one.
    - Justify the identity function, $y = x$, being the line of reflection for a function and its inverse.
- **Other Assessments**
    - Have students complete the following Khan Academy practice exercises on Inverse Functions:
        - "Evaluate Inverse Functions"
        - "Find Inverse Functions"
        - "Verifying Inverse Functions"

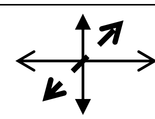
Extensions and Connections

- Have students find the inverse of $y = f(x) = x^2$. Because $f(x)$ is not one-to-one on its domain, restrict the domain to $x \geq 0$.
- Have students graph $y = 10^x$ and $y = \log x$. Ask, *"Are both functions one-to-one? Are they mirror images of one another? If so, with respect to which line? What can we conclude about the two functions?"*

- Pose the following problem: "We have studied many functions, including absolute value, quadratic, square root, and cube root. Of those functions, identify which have inverses and which do not. Explain your reasoning."

Strategies for Differentiation

- Have students play a matching game to match inverse functions, both as graphs and as algebraic expressions. Use cards or an interactive whiteboard.
- Use vocabulary cards for related vocabulary listed above.
- Use the following graphic organizer to remind students of the key concepts and processes for inverses.

$\in$	
$1-1$	$f \circ f^{-1}$
SWAP	

Go over the meaning of the above, as follows:

Elements of the inverse of a function are determined by $(a,b) \in f \Rightarrow (b,a) \in \text{INV of } f$	The graph of a function and its inverse are symmetric to the line $y = x$.
The inverse of f is a function only if f is a one-to-one function.	To prove two functions are inverses of each other, show their composition is the identity function: $f \circ f^{-1}(x) = x$
To find the inverse of a function, **SWAP** x and y, and solve for y.	

Note: The following pages are intended for classroom use for students as a visual aid to learning.

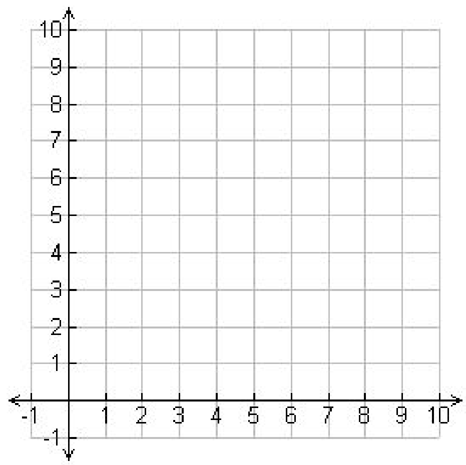
Exploring Inverse Functions

1. Complete the table below showing the relationship between the length of the sides of a square and the area.

Length of Sides	Area
1	
2	
3	

Area	Length of Sides
1	
4	
9	

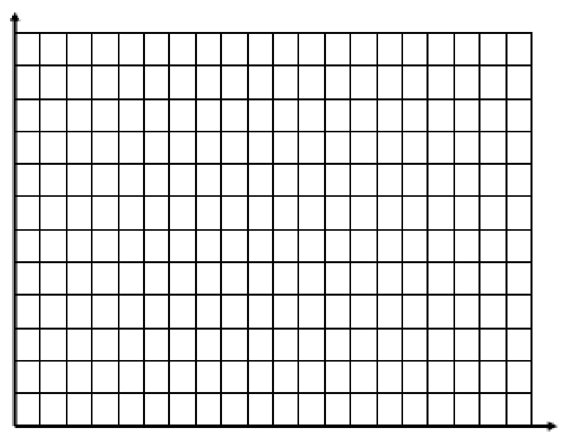
Graph the data in the table on the same set of axis using two different colors of pen to draw each graph.



Answer the following questions:

- How do you find the area of the square given the length of sides?
- How do you find the length of sides given the area?
- Draw a line $y = x$ on the graph above. How are the graphs of the two functions related with respect to the line= x ?

2. The table below shows the dollar-to-peso conversion.



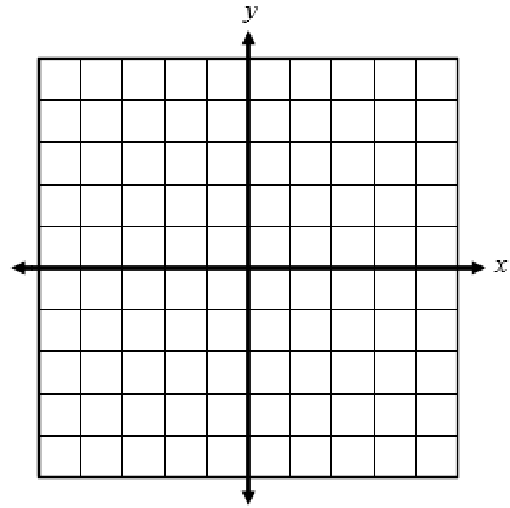
Dollar	**Peso**
10	500
20	1000
50	2500
100	5000

- Draw a graph using the data from the table above.
- Create a formula to convert dollars to pesos.
- How do you convert peso back to dollar?
- Using the same axis, draw a graph with pesos as the domain and dollars as the range.
- Graph $y = x$.

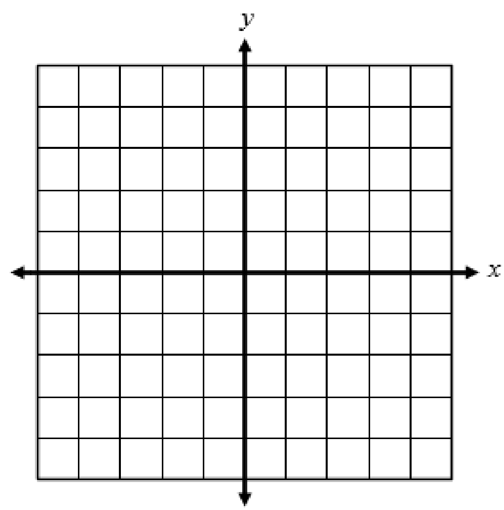
Finding the Inverse of a Function

Find the inverse of each function algebraically. Graph the function and its inverse. Determine whether the inverse is a function. Determine the domain and range of the function and its inverse.

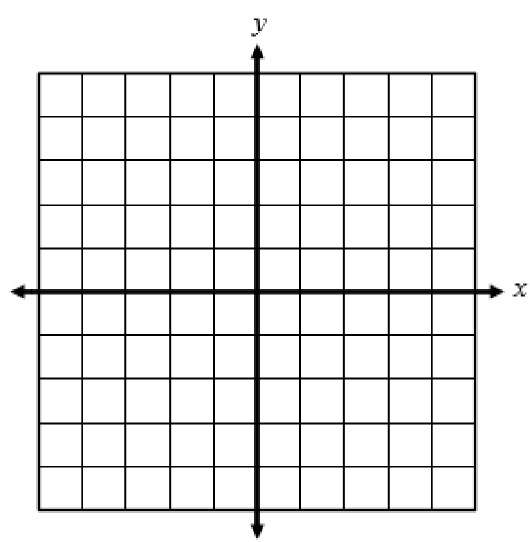
1. $f(x) = \frac{1}{2}x + 6$



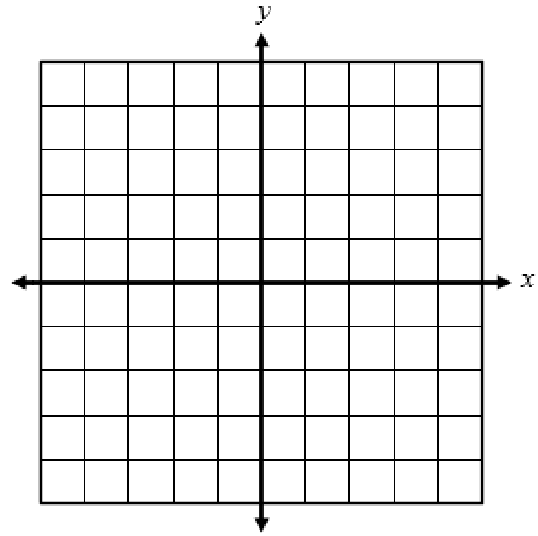
2. $f(x) = (x + 2)^2 - 3,\ x \geq -2$



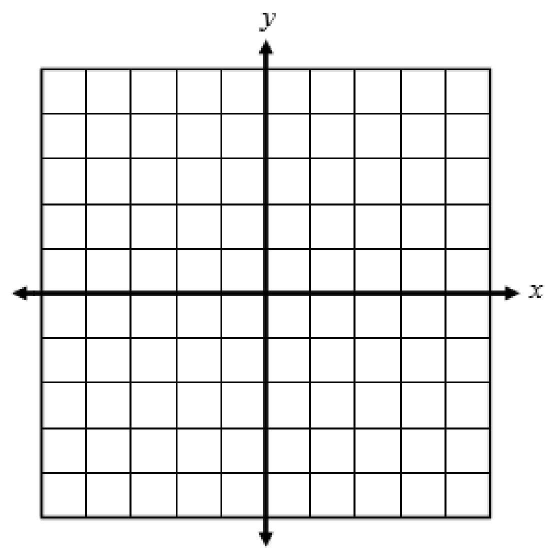
3. $f(x) = 2x^3 + 3$



4. $f(x) = \sqrt{x-3} + 1$



5. $f(x) = \sqrt[3]{x} + 1$



Given the graphs below, graph the inverse of the function. Explain whether the inverse is a function or not. Write the domain and range of the given graph and its inverse.