Rational Equations

Strand: Equations and Inequalities

Topic: Solving equations containing rational algebraic expressions

Primary SOL: AII.3 The student will solve
c) equations containing rational algebraic expressions

Related SOL: AII.1a, AII.1c, AII.3b

Materials

- Solving Rational Equations (Introductory Exercise) activity sheet (attached)
- Steps for Solving Rational Equations Algebraically activity sheet (attached)
- Solving Rational Equations Graphically activity sheet (attached)
- Solving Rational Equations Practice Problems activity sheet (attached)
- Graphing utility

Vocabulary

array, denominator, extraneous solution, factor, least common multiple, numerator, rational algebraic equations, solution

Student/Teacher Actions: What should students be doing? What should teachers be doing?

Time: 60 minutes

1. Distribute the Solving Rational Equations (Introductory Exercise) activity sheet, and have students complete it, working individually and then in pairs to share, confirm, and/or revise their responses. Follow with a class discussion of each problem.

Time: 60 minutes

2. To explore the algebraic and graphical methods for solving rational expressions, begin with the algebraic. Distribute the Steps for Solving Rational Equations Algebraically activity sheet. Encourage students to work with their partners to monitor and communicate what is happening as you lead them through the examples. After working through each example, have student pairs come up and work the similar, accompanying problem. The variety of problems is meant to encompass the scope of typical Algebra II problems.

Time: 90 minutes

3. Distribute the Solving Rational Equations Graphically activity sheet and lead students through the steps in each example, using a graphing utility or direct students to a digital format of activity at the Solving Rational Equations website.
4. Define *extraneous solution* as a result that is not a solution to an equation even though it was obtained by correctly using an equation-solving algorithm. In rational equations, extraneous solutions always result in a division-by-zero error.

Assessment

- **Questions**
    - How do you solve a rational equation algebraically? Explain.

- How can we use the graph of a rational function to find the solution to a rational equation involving that function?
- Given the graph of two rational functions, how can we find the solution to a rational equation where the two functions are equal to each other?

- **Journal/writing prompts**
  - Explain how you can use a graph's points of intersection to solve a rational equation.
  - In your own words, explain what is meant by the term *extraneous solution.* Is it a solution or not? Explain why.
  - Your teacher gave you the rational function $f(x)=\frac{x+2}{x-3}$. Explain whether the function can be equal to 3. Why or why not? Provide two pieces of evidence to support your answer.
- **Other Assessments**
  - Give students solution sets, and ask them to create matching rational equations. (Note: Such open-ended problems allow students to be creative and differentiate the task based upon their own level of understanding.)

Extensions and Connections

- Guide students to make connections to graphing rational functions, paying particular attention to restrictions on the domain and vertical asymptotes.
- Have students complete the Solving Rational Equations Practice Problems activity sheet.
- Introduce equations with complex algebraic fractions.

Strategies for Differentiation

- Use vocabulary cards for related vocabulary listed above.
- Have students solve rational equations by expressing each side as a single fraction first.

Note: The following pages are intended for classroom use for students as a visual aid to learning.

Solving Rational Equations
(Introductory Exercise)

Determine which of the following is true and justify your answers.

1. $x=2$ is a solution to $\frac{3x+6}{x-1}=\frac{x^2+8}{x^2-3}$.

2. $x=-3$ is a solution to $\frac{-2x+1}{1-x}=\frac{3x-2}{4}$.

3. $x=4$ is a solution to $\frac{5x-4}{2x-8}=\frac{20}{x^2-16}$.

4. The solution set for the equation $\frac{2}{x-3}+\frac{5}{x-2}=\frac{7x-19}{x^2-5x+6}$ is all real numbers except 3 and 2. Give two justifications for your answer.

5. The equation $\frac{x^2-10}{x+1}=\frac{-3x}{x+1}$ has exactly two solutions. (Hint: Determine the answer by using your graphing utility to graph $y=\frac{x^2-10}{x+1}$ and $y=\frac{-3x}{x+1}$.)

Steps for Solving Rational Equations Algebraically

Example 1: $\frac{2x}{3}+\frac{x-2}{5}=\frac{1}{6}$

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case, the LCM of 3, 5, and 6 is 30.

$$30\times\left(\frac{2x}{3}+\frac{x-2}{5}\right)=\left(\frac{1}{6}\right)\times 30 \quad \Rightarrow 30\left(\frac{2x}{3}\right)+30\left(\frac{x-2}{5}\right)=5$$

Step 2: Simplify and solve the familiar equation.

$$\Rightarrow\ 20x+6(x-2)=5$$

$$\Rightarrow\ 26x-12=5$$

$$\Rightarrow\ 26x=17$$

$$\Rightarrow\ x=\frac{17}{26}$$

Step 3: Verify the solution.

Does $\frac{2\left(\frac{17}{26}\right)}{3}+\frac{\left(\frac{17}{26}\right)-2}{5}=\frac{1}{6}$?

Problem 1: Now, you follow the steps to solve $\frac{x+1}{2}-\frac{5x}{3}=-x$.

Example 2: $\dfrac{4}{x+2}+\dfrac{5}{x-2}=\dfrac{29}{x^2-4}$

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case, the LCM of $x + 2$, $x - 2$, and $x^2 - 4$ is $(x + 2)(x - 2)$.

$$(x+2)(x-2)\left(\frac{4}{x+2}+\frac{5}{x-2}\right)=\left(\frac{29}{x^2-4}\right)(x+2)(x-2)$$

$$\Rightarrow (x+2)(x-2)\left(\frac{4}{x+2}\right)+(x+2)(x-2)\left(\frac{5}{x-2}\right)=\left(\frac{29}{x^2-4}\right)(x+2)(x-2)$$

Step 2: Simplify and solve the familiar equation.

$$\Rightarrow \cancel{(x+2)}(x-2)\left(\frac{4}{\cancel{x+2}}\right)+(x+2)\cancel{(x-2)}\left(\frac{5}{\cancel{x-2}}\right)=\left(\frac{29}{\cancel{x^2-4}}\right)\cancel{(x+2)(x-2)}$$

$$\Rightarrow \ 4x-8+5x+10=29$$

$$\Rightarrow \ 9x+2=29$$

$$\Rightarrow \ 9x=27$$

$$\Rightarrow \ x=3$$

Step 3: Verify the solution.

Does $\dfrac{4}{x+2}+\dfrac{5}{x-2}=\dfrac{29}{x^2-4}$ when $x=3$?

Problem 2: Now, you follow the steps to solve $\dfrac{7}{x+2}-\dfrac{4}{x-3}=\dfrac{-14}{x^2-x-6}$.

Example 3: $\dfrac{x}{x-1}+\dfrac{3}{x}=\dfrac{5}{2x}$

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case the LCM of $x-1,\ x,$ and $2x$ is $2x(x-1)$.

$$\Rightarrow 2x\cancel{(x-1)}\left(\frac{x}{\cancel{x-1}}\right)+2\cancel{x}(x-1)\left(\frac{3}{\cancel{x}}\right)=\left(\frac{5}{\cancel{2x}}\right)\cancel{2x}(x-1)$$

Step 2: Simplify and solve the familiar equation.

$$\Rightarrow 2x^2+6x-6=5x-5$$

$$\Rightarrow 2x^2+x-1=0$$

$$\Rightarrow (2x-1)(x+1)=0$$

$$\Rightarrow x=\frac{1}{2} \text{ or } x=-1$$

Step 3: Verify the solution.

Does $\dfrac{x}{x-1}+\dfrac{3}{x}=\dfrac{5}{2x}$ when $x=\dfrac{1}{2}$? $x=-1$?

Problem 3: Now, you follow the steps to solve $\dfrac{x}{x+1}=\dfrac{5}{2x-2}-\dfrac{1}{2}$.

Example 4: $\dfrac{x+3}{x+2}=1-\dfrac{x+1}{x+2}$

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case the LCM is just $x + 2$.

$$\Rightarrow\ (x+2)\left(\frac{x+3}{x+2}\right)=\left(1-\frac{x+1}{x+2}\right)(x+2)$$

Step 2: Simplify and solve the familiar equation.

$$\Rightarrow x+3=(x+2)-(x+1)$$

$$\Rightarrow\ x+3=1$$

$$\Rightarrow\ x=-2$$

Step 3: Verify the solution.

When we check $x = -2$, we can see that it leads to division by zero. Because we correctly followed the process for finding a solution and the result we generated does not solve the equation, it is call an extraneous solution. Thus, the answer to the problem is “no solutions.”

Problem 4: Now, you follow the steps to solve $\dfrac{4x}{x+1}+\dfrac{x+5}{x+1}=3$.

Example 5: $\frac{x+3}{x+2}=2-\frac{x+1}{x+2}$ (note how similar this is to Example 4.)

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case, the LCM is just $x+2$.

$$\Rightarrow (x+2)\left(\frac{x+3}{x+2}\right)=\left(2-\frac{x+1}{x+2}\right)(x+2)$$

Step 2: Simplify and solve the familiar equation.

$$\Rightarrow x+3=(2x+4)-(x+1)$$

$$\Rightarrow x+3=x+3$$

$\Rightarrow x$ can be any real number and satisfy $x+3=x+3$, but when we want to solve $\frac{x+3}{x+2}=2-\frac{x+1}{x+2}$, x cannot equal −2. Why?

So, the solution set is all real numbers except −2, or in set-builder notation, $\{x\,|\,x\neq -2\}$.

Step 3: Verify solution.

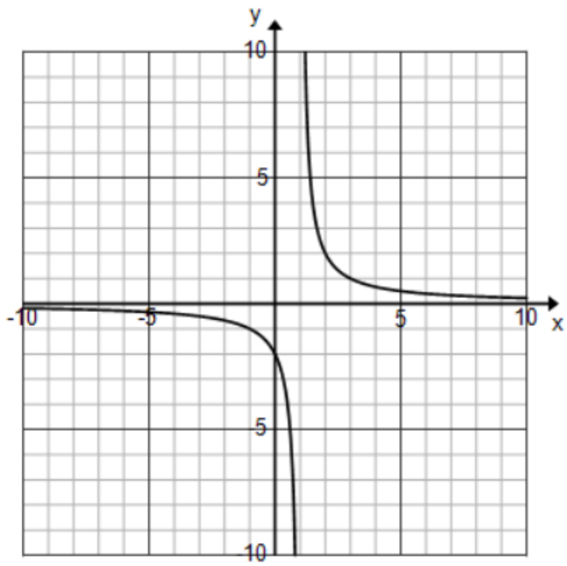
It is impossible to check all real numbers as solutions, but verify a couple**.**

Problem 5: Now, you follow the steps to solve $\frac{4x}{x+1}+\frac{x+5}{x+1}=5$.

Solving Rational Equations Graphically

The link to this activity is https://tinyurl.com/rationalequations

Use the graph of the function $f(x)=\dfrac{2}{x-1}$ below to complete each of the following.

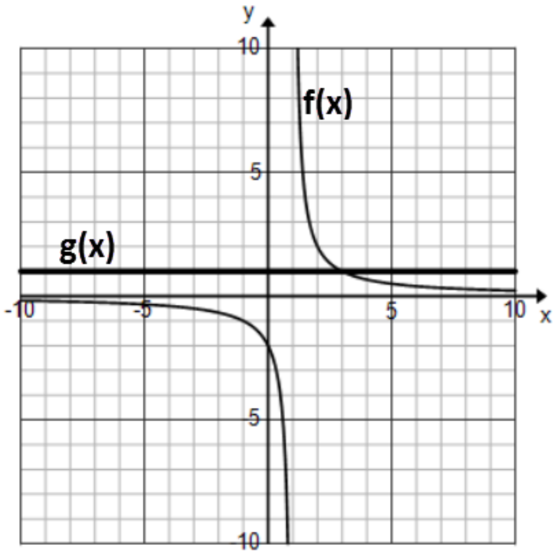


1. List at least three things you notice about the function.

2. Evaluate each of the following. Give at least two justifications to support your answers.

   a) $f(-1)$ b) $f(0)$ c) $f(3)$

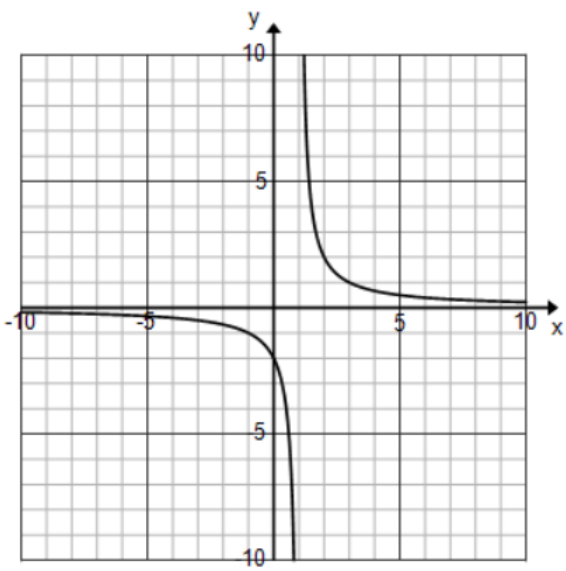
3. What is the value of x when $f(x)=1$? Give two justifications for your answer.



4. The functions $f(x)=\frac{2}{x-1}$ and $g(x)=1$ are graphed at the left. Where do the graphs intersect?

5. What do you notice about your answers to problems 2c and 3, and your answer in 4?

6. Is $x=3$ a solution to the equation $\frac{2}{x-1}=1$? Justify your answer.



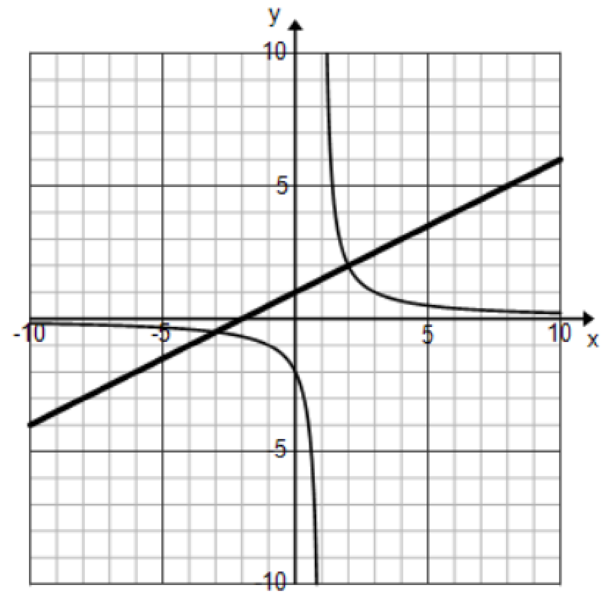
7. When does $f(x)=\frac{2}{x-1}$ equal –4? Give two justifications for your answer.

8. Is $x=\frac{1}{2}$ a solution to the equation $\frac{2}{x-1}=-4$? Why or why not?

9. Can $f(x)=\frac{2}{x-1}$ equal 0? Why or why not? Give at least two justifications for your answer.

10. Jeremy is solving the rational equation, $\frac{2}{x-1}=\frac{x+2}{2}$. He says that $x=-3$ is one solution to the equation. Is he correct or incorrect? Explain and justify your answer.

The functions $f(x)=\dfrac{2}{x-1}$ and $g(x)=\dfrac{x+2}{2}$ are graphed on the right.



11. What do you notice about the graphs?

12. Where do they intersect?

13. What is another solution for the rational equation $\dfrac{2}{x-1}=\dfrac{x+2}{2}$? Justify your answer.

How can we solve a rational equation by graphing it?

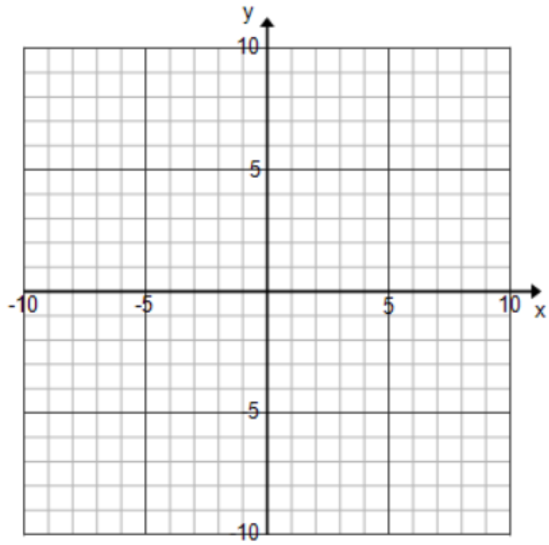
14. Oliver is solving the rational equation $\dfrac{2}{x-1}=\dfrac{x}{x^2+x-2}$. He says that both $x=-4$ and $x=1$ can be solutions to the equation. Anya says that he's partially correct, that only $x=-4$ is a solution. Who is correct? How do you know?

15. Anya hasn't seen the graph of the equation. How did she know that $x=1$ wasn't a solution?

16. What values of x cannot be solutions to the rational equation below? Why?

$$\frac{1}{x-5}+\frac{1}{x-5}=\frac{4}{x^2-25}$$

17. Solve the rational equation $\frac{x-2}{x+3}-\frac{1}{x-2}=\frac{1}{x^2+x-6}$. Sketch a graph below to support your answer.



Solving Rational Equations
Practice Problems

Solve each of the following. Be sure to check solutions in the original equations and identify any extraneous solutions.

1. $\frac{4}{x} + \frac{1}{3x} = 9$

2. $\frac{3}{n+1} = \frac{5}{n-3}$

3. $\frac{2}{x+5} - \frac{3}{x-4} = \frac{6}{x}$

4. $\frac{1}{x-5} + \frac{1}{x-5} = \frac{4}{x^2-25}$

5. $\frac{6x^2+5x-11}{3x+2} = \frac{2x-5}{5}$

6. $\frac{3}{x-1} - \frac{4}{x-2} = \frac{2}{x+1}$

7. $\frac{x}{x^2-4x-12} = \frac{x+1}{6-x} - \frac{x-3}{2+x}$

8. $\frac{c+2}{c-5} = \frac{7}{c+2}$

9. $\frac{x^2-2x-3}{x^2-x-6} - \frac{x}{x+2} = \frac{5-x}{x-3}$

Solve each of the following. Include all work associated with the solving process.

10. A number minus 5 times its reciprocal is 7. Find the number.

11. A swimming pool can be filled in 12 hours using a large pipe alone and in 18 hours with a small pipe alone.
    - If both pipes are used, how long will it take to fill the pool?
    - If the fire department is asked to help out by using its tank, which can fill the pool in 15 hours, how long will it take to fill the pool if all three water sources are used simultaneously?
    - At the end of the summer a drain is left open, and the pool empties in 24 hours. If the pool attendant who is preparing the pool for opening day forgets to plug the drain, can the pool be filled in one day? If so, how long will it take? If not, explain why not.

12. John is deciding between two jobs. One promises convenient hours and is closer to his home. The other pays $2.25 more per hour, and John would earn $1,000 in 10 hours less than it will take him to earn $900 at the first job. What is the hourly wage for each job?

13. The current in a river is estimated to be 4 mph. A speedboat goes downstream 6 miles and comes back 6 miles in 15 minutes. What is the average speed of the speedboat in still water? (Hint: Let r be the average speed in mph in still water.)

14. Find all real numbers x, $x + 2$, and $x + 4$, such that the reciprocal of the smallest number is the sum of the reciprocals of the other two. Are there rational solutions to this problem?

15. A manufacturing company has a machine that produces 120 parts per hour. When a new machine is delivered that makes twice as many parts per hour, how long will it take the two machines working together to make 120 parts per hour? (Hint: Let x be the time [number of hours] the machines take to make the 120 parts.)