REPRESENTATIONS & RAFTS

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Building a rod raft allows students to make mathematical connections among a model, a table, a formula, and a graph.

To build on prior knowledge and mathematical understanding, middle school students need to be given the opportunity to make connections among a variety of representations. Graphs, tables, algebraic formulas, and models are just a few examples of representations that can help students explore quantitative relationships. As a mathematics educator, I conduct workshops and teach courses for middle school teachers. By experiencing the problem-solving process themselves, these teachers can help their own students make the aforementioned connections.

Smith and Stein (1998) confirm that learning gains are related to the extent to which tasks are set up and implemented to maintain higher-level cognitive demand. The teacher's role in the problem-solving process is key to engaging students in high levels of cognitive thinking and reasoning. One way to accomplish this is to use open-response problems that require students to struggle with making mathematical concepts fit within their own solution plan. In so doing, they will be better able to form a conceptual understanding of the mathematics involved in the problem.

Students often have preconceived ideas or even misconceptions and may know a list of steps or algorithms. Although they can perform these calculations, they may not always understand the concept or mathematical connections. This article shows how a problem-solving task can be used to help students make sense of mathematics, construct meaning, and make algebraic connections between existing and new mathematical ideas.

I use one open-response problem,

called the Raft Maker Task, with middle school students and both preservice and in-service teachers. This task was adapted from Building Rafts with Rods (Annenberg Media 1997).

SYNOPSIS OF THE RAFT MAKER TASK

This problem asks students to find the surface area and volume for thirteen different-size rafts (see the instructions in **fig. 1**). Students initially use Cuisenaire Rods® to build one-color rectangular prisms (rafts) and record data. In their various constructions, students will make anywhere from 1 to 10 rafts, using the same color rods for each raft. Cuisenaire Rods come in ten different colors, ranging in length from the 1 cm white rod to the 10 cm orange rod. The purple rod is a 4 cm × 1 cm × 1 cm rectangular prism. Used alone, it would constitute the first purple raft.

Students construct the second raft by combining two purple Cuisenaire Rods to create a 4 cm × 2 cm × 1 cm rectangular prism. The third purple raft, a 4 cm × 3 cm × 1 cm rectangular prism, consists of three purple rods. For each raft in the series, students determine the surface area and volume. **Figure 2** shows the fourth raft. Four purple rods, which create a rectangular prism that is 4 cm × 4 cm × 1 cm, have a surface area of 48 cm^2 and a volume of 16 cm^3.

Fig. 1 Students receive these instructions for the Raft Maker Task.

1. Create all possible one-color rafts for your given Cuisenaire Rod.
2. Create a table to show the surface area and the volume for rafts 1 through 10, raft 12, raft 25, and raft *n*.
3. Create graphs to represent the surface area and volume.
4. Write a good mathematical explanation using appropriate math vocabulary (explain what and why).

Fig. 2 The surface area and volume of the fourth raft are calculated.

How do we measure surface area?
- We notice the number of square centimeters that cover all surfaces.
- Rectangular prisms have six rectangular faces.

How do we measure volume?
- How much three-dimensional space is in the rectangular prism?
- What is the number of cubic centimeters?

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As students determine surface area and volume, they record their results in a table and look for patterns; students then use these patterns to complete the table, indicating the surface area and volume of the remaining, unbuilt rafts. In addition, they are challenged to develop the explicit formula for the surface area and volume of the *n*th raft. Students then create a graph that represents the data in their table. Ultimately, students are encouraged to make connections among their pattern(s), formula, table, and model. As students complete this process, they use mathematical vocabulary to explain in writing what they did and why they did it.

Some teachers are skeptical of their students' ability to solve such complex problems and make mathematical connections without being taught specific procedures. One teacher said, "My students can't do this—I don't think they could find the *n*th term or develop a formula." However, this same teacher later shared that her fifth-grade students became excited and responsive when given such tasks and raised their hands to proudly announce, "I know what *n* is! I know what *n* is!" Teachers who implemented the Raft Maker Task have confirmed what I have deduced in my own work with students: Teaching through problem-solving tasks may take longer, but these tasks will help students understand and connect underlying mathematical concepts and relationships.

To aid middle school teachers in replicating this activity and using it as a springboard to design other open-response problem-solving tasks, the remainder of this article will describe the Raft Maker Task through student

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work as well as verbal and written explanations.

MAKING MATHEMATICAL CONNECTIONS

To implement any problem-solving lesson, it is imperative for the teacher to pose the problem succinctly and ensure that students become engaged with the task (Hartweg and Heisler 2007). Within the first five minutes of class, the instructor poses the task, elicits a working definition for a Cuisenaire Rod raft by showing an example of the fourth purple raft, and reviews the meaning of surface area and volume. Students then break into small working groups, receive a specific color of Cuisenaire Rod with which to build their one-color rafts, and begin collecting and analyzing data.

Using the Model, the Table, Verbal Descriptions, and Written Explanations

Some groups will begin work right away, others will review the requirements of the task, and still others will have trouble getting started. The teacher can help the groups that are off task by asking these questions:

- Can you show me the second raft?
- What is the volume of the second raft? How did you find the volume?
- What is the surface area of the second raft? How many surfaces will you need to account for?
- What unit will you use to measure the surface area? Why?
- How many square centimeters are on the top?

After ensuring that all groups are on task, the teacher should continue to move from group to group to assess understanding while also giving students the opportunity to verbalize their mathematical thinking and relate their models to the mathematics.

Teaching mathematics through problem solving is designed to increase students' ability to think and reason. However, teachers may need to use questioning techniques to help students make connections between verbalizing their thinking processes and writing their explanation. While working through the Raft Maker Task, teachers will note that students often confuse surface area and volume or have difficulty understanding the units they are measuring. When questioned, students tend to explain surface area and volume in terms of how to calculate, offering explanations such as, "Take the length times the width for all the sides and then add them together," and "Take the base times the height," rather than explaining the attribute of what is actually being measured.

Middle school students also have difficulty determining the surface area, but their understanding can improve as they use centimeter grid paper to determine the number of squares covering all six surfaces. In most cases, students use the grid paper while working on the first few rafts but eventually choose to mentally calculate the areas for each face. They then add the faces for subsequent rafts.

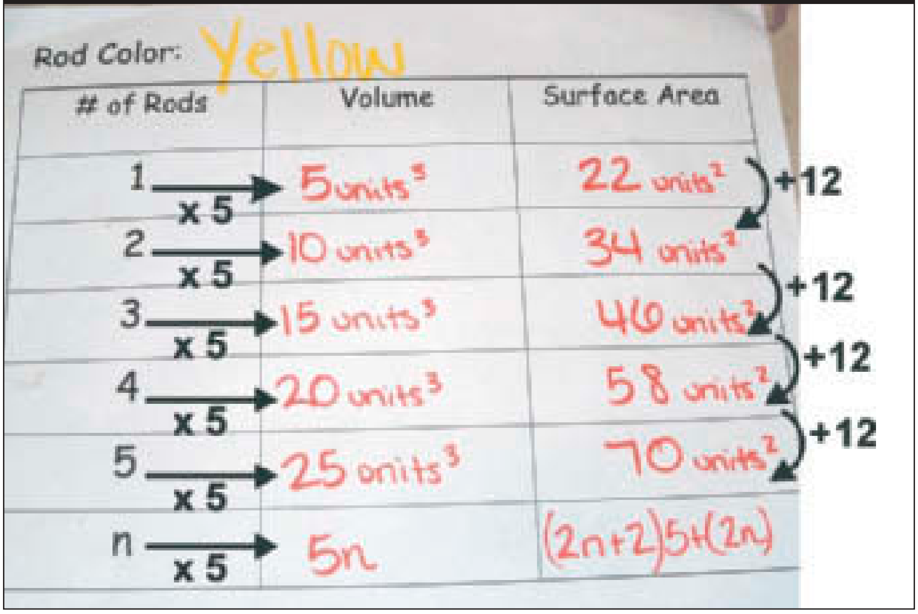
After students find the surface area and volume for the first few rafts, they begin to look for patterns. Students usually see a multiplicative relationship between the number of Cuisenaire Rods used for each raft and the volume of each raft and are able to complete this part of the table rather quickly. At this point, students can determine the volume for any raft. The teacher can provide various inputs or number of rods for students to state the outputs or volume.

In **figure 3**, for example, the volume for the given number of rods is easy for students to determine by an *explicit rule*: They multiply the number of rods by 5. This student explains how the process of determining the volume had to be "acted out" using the Cuisenaire Rod models in a written explanation.

Using Models and Tables to Build Explicit and Recursive Rules to Represent Functions

For the first part of the task, students are able to find the volume of any raft

Fig. 3 A completed table for the yellow-rod rafts demonstrates how the explicit formula for the volume is discovered. Yellow rods are five units long.



Rod Color: Yellow

# of Rods	Volume	Surface Area
1 → x 5	5 units³	22 units²
2 → x 5	10 units³	34 units²
3 → x 5	15 units³	46 units²
4 → x 5	20 units³	58 units²
5 → x 5	25 units³	70 units²
n → x 5	5n	(2n+2)5+(2n)

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Fig. 4 Two students analyze their dark-green Cuisenaire Rod rafts. Dark-green rods are six units long.

(a)

I added a second rod and I knew the length was 6 cm and the width was 2 cm [for the top face], so I multiplied those and my product was 12 cm². Then I knew the top and bottom were equivalent so I multiplied 12 [cm²] by 2. My product is now 24 [cm²]. Next, I knew the sides still equaled 6 cm long, so since there are 2 sides of that length, I multiplied 6 [cm²] × 2, and my product was 12 [cm²]. Then the surface area of the other sides were 2 [cm²], because there are now 2 rafts, so I multiplied 2 [cm²] × 2 because there were 2 of those sides. My product was 4 [cm²]. So I added all my products up, which were 24 [cm²], 12 [cm²], and 4 [cm²]. My final answer for the [surface] area of raft 2 was 40 [cm²]. I did this for all the rafts up to raft number 5.

(b)

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number in a direct, or *explicit*, manner as described above. Generally, we ask students to build explicit rules so that they can use the number of rods (as shown in the first column in **fig. 3**) to determine both the volume and the surface for any size of raft. However, to find the surface area, most students find it easier to recognize and use a step-by-step, or *recursive*, method (Rubenstein 2002). In other words, to find a relationship or pattern for the surface area of the five-unit-long yellow rod, students generally notice a constant difference of 12 cm² between each surface area (as shown in **fig. 3**). Students then use that difference to find the surface areas for each subsequent raft.

To help students make associations and mathematical connections between graphs and linear equations, a teacher can encourage them to relate their numeric pattern discoveries in the table with the actual model. For example, the dark-green rafts that are six units long, used to create the table in **figure 4a**, show a recursive difference of 14 cm² in the surface-area column. In a videotaped dialogue with an eighth-grade prealgebra student, the teacher asks:

> Why is there a difference of 14 cm²? Can you relate it to the model? It would seem to me that if you were adding the extra rod, you should be able to show the 14 cm² on the extra one. Think for a little bit. Where on the new rod is the 14 cm²?

After considerable thinking, the student determines which sides of the new rod did not change the surface areas and which sides account for the extra 14 cm².

Rubenstein (2002) confirms that "because so many students have a natural inclination to look first for recursive rules, finding explicit rules may be hard for them" (p. 248). This scenario proves that point. That is, using the usual table form, students find patterns that are "going down the table" easier to understand than those "going across the table." This is perhaps not surprising, as often the patterns "going down" are mostly additive; the multistep patterns "going across" include multiplicative components.

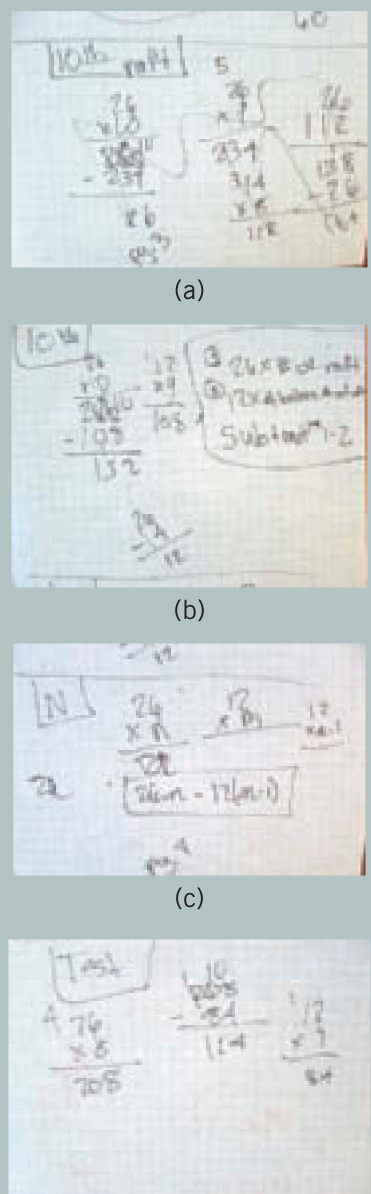
In my work with rafts, students have difficulty finding the explicit rule for the surface area. However, they are successful when guided by teacher questions that prompt them to connect the model and the table while developing the formula:

- To get your surface area for the seventh raft, did you have to keep adding 14 cm^2 to the ones before? Where on the model is that shown?
- It was interesting to me that you could easily find the volume of the 24th raft by taking the raft number times 6 [cm^3] [an explicit rule]. So how would you find the volume for the 100th raft?
- Using the information in your table, what can you do with the raft number to find the correct surface area?
- Something that you're doing is good; see if it works on your 5th or 6th raft. Does the process work for any raft?

One student's struggle with composing and decomposing numbers and expressions from the table is shown in **figure 5**. When the process does not produce the desired outcome, the student crosses off what did not work and begins the process again. Her struggle and persistence pay off when she finds the correct answer (see **fig. 5b**). Instead of explaining the process using "the 10th raft," the teacher asks the student to explain what would happen with *any* raft. As she explains the process, the student writes the steps in words and numbers as shown in the box on the right-hand side in **figure 5b**. While referring to the Cuisenaire Rod model, this student explains her mathematical thinking. To build the explicit rule to represent the function, she draws on the abstraction from working with particular numbers to generate a general expression.

Fifth-grade students proudly announce, "I know what n is! I know what n is!"

Fig. 5 A student develops the explicit rule for the surface area of the tenth and *n*th dark-green rod rafts but struggles along the way.



(a) (b) (c) (d)

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Teachers who have used this Raft Maker Task witness their students' excitement as well as success when they are able to take the recursive information from the table, relate it to the model, and develop the explicit rule. Most students could accomplish this much of the Raft Maker Task within one or two class periods.

To conclude each class period, the teacher asks students to write their mathematical explanations and briefly report their mathematical discoveries to the class. This wrap-up activity helps solidify students' thinking and verify the relationships found among the table, the model, and the explicit formula.

Connect Models, Tables, Formulas, and Graphs

For the next part of the task, students record the information from their rafts in a table that includes any explicit rules (if discovered) for finding the volume and surface area. The nonstandard format can be compared with the standard format in **figures 3** and **4a**, respectively. If students are unable to develop the explicit rule for the surface area, they can leave a table such as **figure 3** blank and continue creating graphs to represent the volume and the surface area for their assigned Cuisenaire Rod rafts. Teachers who do not provide students with specific guidelines for creating graphs generally obtain various representations that can be discussed with the whole class.

Although students in a prealgebra or algebra class may be able to simplify their surface-area rules for n, many middle school students need to check the table information and the explicit rules of others by entering an input number to check the output result. It is interesting to note the discussions that occur among students, particularly when the same rod color produces two seemingly different explicit rules for the surface area.

After students verify the findings from other groups, students should study the various graphs. This is a good time for the teacher to introduce or revisit the meaning of the terms *linear* and *slope* as students compare the relationships among different raft colors. The connections among the slope of a line, the explicit rule, the table information, and the model become clearer as students study their surface-area graph and are asked to do three things:

1. Determine if the graph indicates a linear relationship.
2. Find the slope of a line.
3. Show where that information is represented in the table and on the model.

Fig. 6 Using a standard table format and bar graphs for the brown rafts help students make connections between the recursive and explicit rules. (The missing measurement unit labels can be addressed at a follow-up lesson.)

Rod Color:

# of Rods	Volume	Surface Area
1	8	34
2	16	52
3	24	70
4	32	88
5	40	102
n	n(8)	n(18)+16

(a)
Table

(b)
Graphs

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It is exciting when students "discover" the connections among a table, a formula, and a graph. In so doing, they learn that the recursive method for determining the surface area helps develop the explicit rule and represents the slope of a line represented on a graph.

For students to make a similar connection to the y-intercept, dialogue such as the following may clarify their understanding:

Teacher: What you are looking for is a relationship between the slope and the formula in the graph.

Student: There the slope is 18 [refers to **fig. 6** and the surface-area column in the brown-rod table], and you're multiplying by 18 there [refers to the explicit rule $n \times (18) + 16$] and in the other example, the slope [refers to **fig. 3** and the surface-area column on the yellow-rod table] is 12 and you're multiplying by 12 here [refers to the explicit rule $10 + 12n$].

This student has already recorded her data on the x-axis and y-axis opposite of the graphs found in **figure 6**. This understanding required some reflective thinking on her part to make the connections to her own graph, but she was able to demonstrate her understanding and to make the mathematical connections.

CONCLUSION

Throughout my work with middle school students and teachers, I have found that it is important to give students opportunities to struggle with mathematical ideas, discuss mathematical thinking, and record findings and thought processes in writing. This process not only aids student understanding but also provides insight for the teacher. For example, this project could lead to follow-up lessons in which students learn such mathematical ideas and technicalities as—

1. multiplying by 5 cm^3, not 5;
2. adding missing labels (see **fig. 6a**), which are essential; and
3. representing discrete data (see **fig. 6b**) with points that are not on a line.

The task that teachers choose and the questions that they pose can help students make connections that might otherwise go unnoticed. The Raft Maker Task provides students with a means of using a model to develop recursive and explicit rules and make mathematical connections. The excitement in making these connections using an open-response, problem-solving task is visible when students exclaim, "I know what *n* is!" as well as when teachers report, "This is it—this can tie it all together—a model, a table, a formula, a graph. . . ."

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